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# The Geometry of Consistency: Decohering Histories in Generalized Quantum Theory

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## Abstract

The geometry of decoherence in generalized “consistent histories” quantum theory is explored, revealing properties of the theory that are independent of any particular application of it. Attention is focussed on the case of quantum mechanics in a finite dimensional Hilbert space  $H$ . It is shown how the domain of definition of a general decoherence functional naturally extends to the entire space of linear operators  $\mathcal{H} = \mathcal{L}(H)$  on  $H$ , on which the decoherence functional is an Hermitian form. This identification makes manifest a number of structural properties of decoherence functionals. For example, a bound on the maximum number of histories in a consistent set is determined. In the case of the “canonical” decoherence functional with boundary conditions at the initial and final times, it is shown that the maximum number of histories with non-zero probability in any decohering set is  $r_\alpha r_\omega$ , where  $r_\alpha$  and  $r_\omega$  are the ranks of the initial and final density operators. This is one reason that some coarse graining is generally necessary for decoherence. When the decoherence functional is positive – as in, for instance, conventional quantum mechanics on a Hilbert space  $H_S$ , where the histories of a closed physical system  $S$  are represented by “class operators” in  $\mathcal{H} = \mathcal{L}(H_S)$  – it defines a semi-inner

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product in  $\mathcal{H}$ . This shows that consistent sets of histories are just orthogonal sets in this inner product. It further implies the existence in general of Cauchy-Schwarz and triangle inequalities for positive decoherence functionals. The geometrical significance of the important ILS theorem classifying all possible decoherence functionals is illuminated, and a version of the ILS theorem for decoherence functionals on class operators is given. Additionally, the class of history operators consistent according to a given decoherence functional is found (most explicitly for positive decoherence functionals), and the problem of determining all the decoherence functionals according to which a given set of histories is consistent is addressed. In particular, it is shown how to construct explicitly these decoherence functionals, thus showing how the decoherence functional of a closed quantum system is constrained by observations. Finally, the conditions under which a general decoherence functional on class operators is canonical are determined. More generally, the “geometric” point of view here developed supplies a powerful unified language with which to solve problems in generalized quantum theory.

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## I. INTRODUCTION

This paper studies the geometry of decoherence in finite dimensional Hilbert spaces, with the aim of illuminating the basic formal structures of the quantum mechanics of closed systems. The properties studied are those that arise only from the axioms of the formalism, and hence are universal in the theory, independent of any particular application of it. The framework is that of the generalized quantum theory of closed systems, as first defined by J.B. Hartle in [1,2], and much expanded upon by Isham, Linden, and others [3–8, see [9] for an introduction to their work]. The central structural element of this formalism is the “decoherence functional”, an Hermitian functional on pairs of “histories” of a physical system  $S$ . The decoherence functional both provides a measure of interference between histories, thereby determining which sets of histories are “consistent” in the sense that probabilities may consistently be assigned to the various histories in the set, and fixes those probabilities within consistent (or “decohering”) sets. For closed quantum systems (such as the universe), in which there are no external classical observers around to measure anything, the decoherence functional thus replaces the usual rule of “Copenhagen” quantum

mechanics that probabilities may only be assigned to histories which are measured [1,10].<sup>1</sup> The decoherence functional is a natural generalization to closed systems of the notion of “quantum state”, as the term is used in quantum logic and in algebraic quantization [4], a connection that will be described in section II A.

In this investigation, consideration is restricted to systems whose observables live on a finite dimensional Hilbert space  $H_S$ . The question of decoherence is approached from the point of view that the alternative histories of the system  $S$  can be taken as living in  $\mathcal{H} = \mathcal{L}(H_S)$ , the linear operators on  $H_S$ , and that decoherence functionals define Hermitian forms on  $\mathcal{H}$  [8]. Indeed, a wide class of decoherence functionals may be thought of as positive (but not necessarily positive definite) inner products on this vector space; these include all of those used in practical applications of generalized quantum theory. Decohering, or consistent, histories are then orthogonal vectors under this inner product, and the “length” (squared) of a history consistent in this inner product is its probability. The study of the “geometry of decoherence” is thus the study of the Hermitian forms and semi-inner products on  $\mathcal{H}$ . The canonical decoherence functionals over  $H_S$ , which correspond to situations in which boundary conditions are imposed on  $S$  at initial and possibly also at final times, are examples of such positive decoherence functionals of particular importance in theories with an externally supplied notion of time; they are the decoherence functionals of ordinary quantum mechanics.

(A parallel treatment of these ideas is also given for the case where quantum histories are represented by projection operators in a bigger Hilbert space  $\otimes^k H_S$ , as in the quantum-logical formulation of Isham *et al.*.)

The focus of this work is thus the *structure* of generalized quantum theory. Particular physical applications of it are not addressed. Rather, I aim to reveal general and universal properties of the formalism that apply to any particular application of it. Apart from the general structural results, the physical problem lurking in the background is the issue of what additional physical constraints may reasonably be imposed on decoherence functionals, and in particular, of how one infers the decoherence functional of a closed quantum system from experiments; the analogous problem in ordinary quantum mechanics is how to reconstruct from observations the density matrix of a quantum system. This is a subject which will be approached more directly in future work, using the tools developed here. (Indeed, I address the *mathematical* side of this problem in section VI, leaving the difficult physical questions for another venue.)

Taking a close look at what the basic assumptions of the theory do and do not imply supplies information concerning the question of which aspects of generalized quantum theory follow from its basic framework, and which are properties of a specific choice of decoherence functional and method for representing quantum histories. This knowledge in turn illuminates the potential physical significance of the choices that are made, and supplies a clear structure within which to study the implications of any further physical conditions imposed on the theory. In addition, a number of useful calculational tools are developed.

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<sup>1</sup>Though I will be employing the formalism of Gell-Mann and Hartle throughout, it should be noted that similar notions were introduced earlier, but independently, by Griffiths and by Omnès [11,12, see [13] and [14] for recent reviews of their work.].

The “geometric” point of view developed here also has the value of building intuition for the mathematics of decoherence.

A number of useful properties of decoherence functionals (particularly the positive ones, which include the decoherence functional of ordinary quantum mechanics) are derived here in a very general setting. These include a bound on the maximum number of histories in any consistent set, and a Cauchy-Schwarz inequality for positive decoherence functionals. Some of these results were known previously for the special case of the decoherence functional that arises in ordinary quantum mechanics, but the present work illuminates the generality of their geometric origins. I also show how to characterize algebraically the canonical decoherence functionals that are used in conventional applications of generalized quantum theory.

This work is thus complementary in spirit to the sophisticated quantum-logical investigations of Isham, Linden, *et al.* [3–8], in some respects building a bridge between their approach to generalized quantum theory and the more conventional one. Among their most significant results is the important ILS theorem [6,8,15] classifying decoherence functionals, in direct analogy with the Gleason theorem classifying algebraic quantum states in ordinary quantum mechanics. (We will see the ILS theorem emerge quite naturally out of the present formulation of generalized quantum theory, illuminating its geometrical significance. In addition, a version of the ILS theorem for decoherence functionals defined on “class operators” appears.) While the program of Isham and Linden is unquestionably appropriate for the mathematically rigorous formulation of generalized quantum theory, and in particular is important to the program of generalizing quantum mechanics sufficiently to encompass the requirements of quantum gravity and other theories without an *a priori* definition of time, the present work makes more direct contact with the way in which decoherence functionals are actually used in computations, and the elucidation of the geometry of decoherence in  $\mathcal{H}$  has its own intrinsic interest and usefulness.

After these introductory words, the succeeding sections proceed in the following fashion: section II defines the notion of a generalized quantum theory, realizes this notion in a finite dimensional Hilbert space  $H_S$  by exhibiting the canonical decoherence functional  $d_{\alpha\omega}(\cdot, \cdot)$  over  $H_S$ , and discusses quantum histories and their operator representations. Section III explains how to extend the domain of definition of a general decoherence functional on a Hilbert space  $H$  to the entire space of linear operators  $\mathcal{L}(H)$  on the Hilbert space  $H$ , showing that it is an Hermitian form on  $\mathcal{H} = \mathcal{L}(H)$  [8]. (In the quantum-logical approach of Isham and Linden,  $H$  is not the same as  $H_S$ , as will be explained in the sequel.) This observation is then exploited to bound the maximum number of histories in a consistent set, and show how positive decoherence functionals are actually semi-inner products on  $\mathcal{H}$ . Section III D develops some useful tools for calculating with a decoherence functional on  $\mathcal{H}$ . Section IV lists some basic structural issues in the program of generalized quantum theory, the solutions to which are described in the balance of the paper. Section V shows how to find all the histories (operators) consistent according to a given decoherence functional, and section VI addresses the converse problem. Finally, section VII solves the problem of determining when an arbitrary decoherence functional on class operator histories is canonical. Section VIII is reserved for summary and some final discussion.

Appendix B summarizes the important notation for ease of reference.

This is a long document, so I offer occasional suggestions about sections that may be

skipped on a first reading. Those interested in an overview of the “geometric” perspective on generalized quantum theory employed here should read through sections II A, II C, II E, III A, III D, and III F first.

## II. GENERALIZED QUANTUM THEORY

This portion of the paper is devoted to a description of the generalized quantum theory and its Hilbert space implementation that will be the subject of the remainder of the paper. Section II A defines in the abstract the idea of a generalized quantum theory. Section II B describes the realization of this idea in quantum temporal logic, and discusses the ILS theorem which classifies the possible decoherence functionals. Sections II C and II D begin a close look at the representation of quantum histories by operators in Hilbert space, making it possible to define decoherence functionals as operators on those histories. Finally, section II E defines the canonical decoherence functional, the decoherence functional that arises out of conventional quantum theory. The rest of the paper goes on to discuss in more detail the idea of the decoherence functional as an operator on histories.

### A. Generalities

A “generalized quantum mechanics” is defined by Hartle [1,2] to consist in:

I. **Fine-Grained Histories:** Sets of exhaustive collections of alternative physical histories of  $S$ . By definition, a fine-grained history is the most refined description of a history possible. The empty history  $\emptyset$  (or 0) is always a member of a fine grained set. Familiar examples of a “history” include a particle’s path in non-relativistic quantum mechanics, and the spacetime configuration of a quantum field in quantum field theory.<sup>2</sup> It is possible, and indeed common, for there to be many allowed complete sets of fine-grained histories. The democracy of Dirac’s transformation theory in ordinary quantum mechanics is a common illustration of this.

II. **Allowed Coarse-Grainings:** The allowed partitions of a set of fine-grained histories into mutually exclusive and jointly exhaustive classes. Each class of an allowed partition is defined to be a coarse-grained history. The trivial coarse grainings where the partition is merely the identity (no coarse graining at all), and where all of the histories in an exhaustive fine grained set are collected together into the completely coarse-grained history  $u$ , are always taken to be allowed coarse grainings. ( $u$  is assumed to be the common complete coarse graining of all exhaustive fine grained sets. Thus, specifying  $u$  is equivalent to defining what is meant by a “complete” or “exhaustive” set of histories. Depending on the context,  $u$  is often instead denoted 1.)

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<sup>2</sup>Note, however, that no notion of time is required *a priori* to define a “history” of  $S$ , and may indeed emerge only obliquely (or not at all!) from other physical considerations. For instance, the collection of all four-geometries on some set of manifolds might be taken as the set of fine-grained histories in a quantum theory of gravitation.

A coarse-graining of a more finely-grained (exclusive, exhaustive) set of histories  $\mathcal{S} = \{h\}$  is often denoted  $\overline{\mathcal{S}} = \{\overline{h}\}$ . (To be explicit,  $\overline{h}$  denotes the class containing  $h$ .) I will denote the collection of *all* histories, fine or coarse grained, by  $\mathcal{U}$ .

**III. Decoherence Functional:** A functional  $d : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{C}$  possessing the following properties:

1. *Hermiticity*

$$d(h, h') = d(h', h)^* \quad (2.1)$$

2. *Positivity*

$$d(h, h) \geq 0 \quad (2.2)$$

3. *Additivity (“Principle of Superposition”)*

$$d(\overline{h}, \overline{h}') = \sum_{h \in \overline{h}, h' \in \overline{h}'} d(h, h') \quad (2.3)$$

4. *Normalization*

$$\sum_{h, h' \in \mathcal{S}} d(h, h') = 1 \quad (2.4)$$

for all exclusive, exhaustive sets  $\mathcal{S} \subset \mathcal{U}$ .

**IV. Decoherence Condition:** This condition determines the kind and degree of interference between histories that will be tolerated in sets of alternative coarse grained histories in which probabilities may be assigned, *i.e.* determines which sets of histories are “consistent”, or “decohere”.<sup>3</sup> A good decoherence condition must guarantee that probabilities may be assigned consistently in the decohering sets which it defines.

What is the precise meaning of this? The probabilities for various histories in an exhaustive set of mutually decohering histories are defined to be the diagonal elements of the decoherence functional,

$$p(h) = d(h, h). \quad (2.5)$$

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<sup>3</sup>There is some tension in the literature concerning the proper usage of these terms. It is of particular note that the term “decoherence” is becoming entrenched in much of the literature as referring to the *dynamical process* of the decay of the off-diagonal elements of the density matrix in appropriate semi-classical bases [16]. While this notion is not the same as the consistency of a set of histories it is not unrelated either. (See [17] and [18, note added in proof] for discussion.) Though I have some sympathy with the utility of making this distinction, by using these terms interchangeably, I do nothing here to quiet the discussion.



In order to permit interpretation as a probability, these numbers must satisfy the standard Kolmogorov rules: they must be real numbers between 0 and 1, must satisfy  $p(\emptyset) = 0$  and  $p(u) = 1$ , and be additive under coarse-graining,

$$p(\bar{h}) = \sum_{h \in \bar{h}} p(h) \quad (2.6)$$

for any coarse graining  $\bar{\mathcal{S}} = \{\bar{h}\}$  of the exclusive and exhaustive set  $\mathcal{S} = \{h\}$ . In order for the “probabilities” of (2.5) to be physically interpretable as such, the decoherence condition should be sufficient for these rules of probability theory to be satisfied. Histories in decohering sets are said to be “consistent with” the decoherence functional; it will become clear below that there is indeed no difficulty in thinking of *individual* histories as being “consistent”, as well as sets of histories. The collection of histories which decohere according to the decoherence functional  $d$  (*i.e.* are in *some* consistent set) will be denoted  $\mathcal{D}_d$ . The class of exclusive, exhaustive *sets* of histories  $\mathcal{S}_d$  consistent according to  $d$  will be denoted  $\mathcal{C}_d$ . (The structure of these spaces is a recurring theme of this work.)

An appendix summarizing most of the notation is provided at the end.

If *arbitrary* partitions are allowed coarse-grainings, so-called “weak decoherence”

$$\operatorname{Re} d(h, h') = 0, \quad h \neq h' \quad (2.7)$$

is necessary and sufficient for the correct probability sum rules to be satisfied. However, the stronger condition of “medium decoherence”

$$d(h, h') = 0, \quad h \neq h' \quad (2.8)$$

is much more mathematically convenient and I will take it to be the decoherence condition IV unless otherwise specified; (2.8) both implies (2.7), and arises naturally in many physical problems. (Decoherence conditions both stronger and weaker than these may also find application, but will not be considered here. They are discussed for instance in [10–12,17].) Decoherence might not be required to hold exactly, but only to the accuracy to which the probabilities are used. The physical meaning of an approximate probability in generalized quantum theory is discussed in [1], and various mathematical aspects treated in [19–22].

### *The Decoherence Functional as Generalized Quantum State*

As first observed by Isham and Linden in their quantum-logical formulation of generalized quantum theory, the decoherence functional is a natural generalization of the notion of (algebraic) quantum state to closed systems [4]. This may seem an odd perspective upon first encounter, but, after all, a quantum state has physical significance only in virtue of the questions asked of it. That is the reason for the emphasis on  $d$  as a functional of quantum histories.

(In fact, those familiar with algebraic quantization will recognize the great similarity between the defining conditions of a decoherence functional and those of an algebraic quantum state. Given a  $C^*$  algebra of observables  $\mathcal{A}$ , a quantum state is defined to be a linear map  $\omega : \mathcal{A} \rightarrow \mathbb{C}$  which satisfies  $\omega(A^*A) \geq 0 \quad \forall A \in \mathcal{A}$ , and  $\omega(1) = 1$ ; see, for

example [23,24]. Alternately, in quantum logic, a state on, for instance, the space (lattice)  $\mathcal{P}(H)$  of projections over a Hilbert space  $H$  – the projections correspond physically to single time propositions about the system – is defined as a positive, normalized, real valued map which is additive on disjoint projections:  $\sigma(P \oplus Q) = \sigma(P) + \sigma(Q)$  if  $P$  and  $Q$  are disjoint [25–27].)

The new element the decoherence functional brings to the quantum-theoretic picture is an internally consistent identification of those histories which may sensibly be assigned probabilities. This measure is concrete and quantitative, circumventing the need to forge an unambiguously applicable notion of “measurement” (the elusiveness of which is positively legendary [28].) What it does *not* do is supply the ontological significance of these probabilities. This (generalized quantum theory) remains, after all, quantum mechanics [29]. (See [30,31] for entrée into this knotty realm.)

The decoherence functional of a physical system therefore not only determines the sets of histories about which predictions can be made, but also what is the probability of each such physically realizable quantum history. It is therefore very appropriate to view the decoherence functional as the “quantum state” of the system in this generalized sense.

## B. Quantum Temporal Logic and the ILS Theorem<sup>4</sup>

As already mentioned, the definition of a generalized quantum theory given above has been formulated as a rigorous quantum logical structure by Isham and collaborators [3–8, an excellent lightning introduction is [9]]. The role of the space  $\mathcal{U}$  is played by a so-called orthoalgebra  $\mathcal{UP}$  of history propositions. This is merely a space with: a partial ordering (expressing notions of relative fine or coarse graining); a “biggest” and a “smallest” element (1 and 0) such that  $0 \leq P \leq 1 \forall P \in \mathcal{UP}$ ; a negation (or “complementation”) operation  $\neg$  (expressing the history “not  $h$ ”) such that  $\neg(\neg h) = h$ ; a “disjointness” relation  $\perp$  between histories (corresponding to the idea of mutually exclusive histories); and a (commutative, associative) “join” operation  $\oplus$  on *disjoint* histories (expressing the logical notion “ $h$  or  $h'$ ”) for which  $h \perp h'$  implies  $h \oplus h' \in \mathcal{UP}$ . Finally, there are a couple of sensible identities interrelating these relations, namely, that  $\neg h$  is the unique element for which both  $h \perp \neg h$  and  $h \oplus \neg h = 1$ , and that  $h \leq h''$  iff there is some  $h'$  such that  $h'' = h \oplus h'$ . While it is often the case that  $\mathcal{UP}$  has *more* structure than that of an orthoalgebra (*e.g.* that of a lattice [25,26], in which the “join” operation is extended to *all* pairs of elements, and in addition a “meet” operation is introduced which encodes the logical “and” for all pairs of propositions), the orthoalgebraic structure seems to be the minimum one can usefully get away with. ([26] is a straightforward introduction.) It should come as no surprise that on  $\mathcal{P}(H)$ , the lattice of projections on a Hilbert space  $H$ ,  $\neg P = 1 - P$ , that  $P \perp Q$  iff  $PQ = 0$ , and that  $\oplus$  is just the usual operator sum when  $P$  and  $Q$  are disjoint.

A decoherence functional is then defined just as above, namely, as an Hermitian, positive, normalized map  $d : \mathcal{UP} \times \mathcal{UP} \rightarrow \mathbb{C}$  which is bi-additive on disjoint history propositions. Isham, Linden, and Shreckenberg [6] have given a complete classification of the decoherence

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<sup>4</sup>This section may be skipped on a first reading.

functionals when  $\mathcal{UP}$  is the lattice  $\mathcal{P}(H)$  of projections on a Hilbert space  $H$  with  $\dim H > 2$ , and Wright [8] has extended the theorem to an (almost) general von Neumann algebra (the “almost” is directly related to the dimension requirement.) The ILS theorem is the natural generalization to generalized quantum theory of Gleason’s famous theorem [32,33, see [26,34] for discussion and [26] for a glossed proof] classifying the states on Hilbert space. (The Gleason theorem puts algebraic states in one to one correspondence with density matrices on  $H$ , so long as  $\dim H > 2$ .) Indeed, according to the ILS theorem, (bi-continuous) decoherence functionals over  $\mathcal{P}(H)$  are, when  $\dim H > 2$ , in correspondence with operators  $X$  on  $H \otimes H$  via

$$d(h, h') = \text{tr}[h \otimes h' X]. \quad (2.9)$$

In order to fulfill the Hermiticity requirement, the operators  $X$  satisfy  $X^\dagger = M X M$ , where  $M$  is the operator on  $H \otimes H$  which switches vectors,  $M(u \otimes v) = v \otimes u$ , so that for operators  $O$  and  $O'$  on  $H$ ,  $M(O \otimes O')M = O' \otimes O$ . Further,  $X$  is “positive”,  $\text{tr}[P \otimes P X] \geq 0$  on all projections  $P$ , and normalized,  $\text{tr}[X] = 1$ . (Actually, Hermiticity implies that  $\text{tr}[P \otimes P X_I] = 0$ , where  $X = X_R + i X_I$  is the decomposition of  $X$  into Hermitian parts. Thus, the last two requirements need only be imposed on  $X_R$ .) Note that positivity on all projections is *not* sufficient to guarantee that  $\text{tr}[L \otimes L X] \geq 0 \ \forall L \in \mathcal{L}(H)$ . As all operators may be written as a linear combination of some collection of projections (via, for example, its decomposition into self-adjoint pieces), necessary and sufficient for positivity is what Isham [3] has called a “positive kernel” condition,

$$\sum_{i,j} c_i^* c_j \ d(P_i, P_j) \geq 0 \quad (2.10)$$

for all sets of projections  $P_i$  and complex numbers  $c_i$ .

We shall see  $X$  resurface later, in section III E.

Considerable further discussion regarding the notion of a generalized quantum theory may be found in [1–5,9,21].

### **C. Generalized Quantum Theory in Hilbert Space: Histories and Their Representation**

There are two methods of representing quantum histories of a physical system  $S$  as operators in a Hilbert space: as “class operators” on the Hilbert space  $H_S$ , or as projection operators on a larger space  $\otimes^k H_S$ . The resulting collections of operators are called  $\mathcal{R}_C$  and  $\mathcal{R}_P$ . After describing these two strategies for representing histories, I go on to discuss two important categories of permissible coarse grainings, homogeneous and inhomogeneous, and the associated spaces of history operators  $\overline{\mathcal{R}}_C$  and  $\overline{\mathcal{R}}_P$ . Before moving on to discuss the decoherence functional as an Hermitian form on these operator spaces in section III, I briefly address in section II D the problem of reconstructing histories from their operator representatives, and exhibit the decoherence functional that is used in all conventional applications of generalized quantum theory in section II E.

## Quantum Histories

A canonical example of a generalized quantum mechanics is a direct generalization to closed systems of ordinary quantum mechanics in a Hilbert space  $H_S$ . Physical alternatives are described by projection operators  $P_a$  onto ranges  $a$  of eigenvalues of observables  $A$ . Properly, a fine-grained history  $h$  is then a time-ordered sequence of Schrödinger projection operators,

$$h = (P_{\beta_1}^1, \dots, P_{\beta_{k-1}}^{k-1}, P_{\beta_k}^k), \quad (2.11)$$

at, for simplicity, some finite, fixed set of times  $\{t_1 < \dots < t_{k-1} < t_k\}$  called the “temporal support”  $\mathcal{T}_k$  of the fine grained histories. Here, the superscript labels which observable is taken at time  $t_i$ , the subscript labelling the chosen range of eigenvalues of that observable.  $h$  thus corresponds to the history in which  $S$  is in the range of eigenvalues  $\beta_1$  of observable 1 at time  $t_1$ , the range  $\beta_{k-1}$  of observable  $k-1$  at time  $t_{k-1}$ , and so on. Now, we obviously do not want the definition of the theory to depend on  $k$  in any essential way. As simply setting  $P_{\beta_i}^i$  always to 1 effectively reduces  $k$  to  $k-1$ ,  $k$  can just be imagined to be bigger than any conceivably required value. ([3–5] begin to treat the case where  $\mathcal{T}$  is a segment of  $\mathbb{R}$ . I shan’t deal with that complication here; so long as  $k$  is finite, the expressions below will be mathematically well defined. From a practical point of view, unless it is actually desired to consider alternatives at a continuous sequence of times, merely imagining  $k$  to be as large as necessary is equivalent anyhow.)

In accord with the customary equity of transformation theory in conventional Hilbert space quantum mechanics, *all* projections on  $H_S$  will be allowed choices at each time  $t_i$ .

The notion of disjointness should correspond to the physical idea that if  $h$  is realized, then it would be contradictory to say that  $h'$  is realized also. Therefore two histories (sequences of projection operators) will be said to be *disjoint* if, at one of the times  $t_i$ ,  $P_{\beta_i}^i P_{\beta'_i}^i = 0$ . Similarly, a collection of mutually exclusive (*i.e.* pairwise disjoint) histories will be said to be *exhaustive* if, at each  $t_i$ , the set of projections  $\{P_{\beta_i}^i\}$  taken from each history constitute a complete set,  $\sum_{\beta_i} P_{\beta_i}^i = 1$  (thereby exhausting all possibilities: it can be said with certainty that *something* “happened” at  $t_i$ .)

### Representation of Quantum Histories by Projection Operators

For the purposes of quantum logic, the best way [3] to represent the history proposition  $(P_{\beta_1}^1, \dots, P_{\beta_{k-1}}^{k-1}, P_{\beta_k}^k)$  is as the tensor product

$$P_{\beta_1}^1 \otimes \dots \otimes P_{\beta_{k-1}}^{k-1} \otimes P_{\beta_k}^k. \quad (2.12)$$

This is desirable because these operators on  $\otimes^k H_S$  are in one-to-one correspondence with the  $k$ -fold sequences of projection operators on  $H_S$ . (The rigorous definition of time-ordered products over a possibly continuous sequence of times is treated in [3–5]. This complication will not be considered here.) Two such history representatives are obviously disjoint iff their operator product on  $\otimes^k H_S$  is 0. A set of them is exhaustive if

$$\sum_{\beta_1 \dots \beta_k} P_{\beta_1}^1 \otimes \dots \otimes P_{\beta_{k-1}}^{k-1} \otimes P_{\beta_k}^k = \otimes^k 1. \quad (2.13)$$

(The fully coarse grained history  $u$  is thereby represented by  $\otimes^k 1$ , where 1 is the unit operator on  $H_S$ .) An exhaustive, exclusive set of histories is thus a partition of unity in  $\otimes^k H_S$ .

### *Representation of Quantum Histories by Class Operators*

On the other hand, in everyday quantum mechanics a natural representative of the history  $h$  is the so-called “class operator” on  $H_S$  given by

$$C_\beta = P_{\beta_1}^1(t_1) \cdots P_{\beta_{k-1}}^{k-1}(t_{k-1}) P_{\beta_k}^k(t_k). \quad (2.14)$$

(Note the choice of time ordering is the reverse of that generally preferred by Gell-Mann and Hartle, but it is much more convenient for present purposes.) Unfortunately these are *not* in one-to-one correspondence with sequences of projections, as simple dimensional counting is sufficient to reveal: with  $\dim H_S = N$ , there are  $N^k$  linearly independent operators on  $H_S$  of the form (2.12), but only  $N^2$  linearly independent class operators (2.14).<sup>5</sup> Correspondingly, there is no simple reflection in  $\mathcal{L}(H_S)$  of the disjointness criterion in  $\mathcal{U}$ , to which we must resort. To emphasize the point, given *only* the two class operators  $C_\beta$  and  $C_{\beta'}$ , *there is no way in general to tell* whether they correspond to physically disjoint histories. (Related issues are taken up in section IID.) However, it *is* obviously true that a mutually disjoint set  $\mathcal{S}$  is exhaustive iff  $\sum_\beta C_\beta = 1$ . (Thus,  $u$  is represented by the unit operator on  $H_S$ .) It is not strictly correct to say that  $\{C_\beta\}$  then constitutes a partition of unity in  $\mathcal{L}(H_S)$  because two logically disjoint histories are not necessarily linearly independent as operators. However, as I will show in section IIIB, if the set  $\{C_\beta\}$  is *consistent* according to some decoherence functional  $d$  (*i.e.*  $\{C_\beta\} \in \mathcal{C}_d$ ), then all the  $C_\beta$  with non-zero probability must be linearly independent in  $\mathcal{L}(H_S)$ . Suppose therefore that all linearly dependent subsets of  $\{C_\beta\}$  are coarse grained down to one history. In that case, then, a consistent  $\{C_\beta\}$  is a genuine partition of the unit operator on  $H_S$ .

The origin of the representation of  $\mathcal{U}$  by class operators can be most easily understood by considering  $C_\beta^\dagger$  applied to some pure state  $|\psi\rangle$ ,  $C_\beta^\dagger|\psi\rangle$ . Writing it out, it is easy to see that up to normalization, this object represents a state  $|\psi\rangle$  that evolves unitarily to time  $t_1$ , at which alternative  $P_{\beta_1}^1$  is realized, and so on up to time  $t_k$ . Thus this object expresses the two forms of dynamical evolution that quantum states in ordinary “Copenhagen” quantum mechanics can undergo, wave function collapse upon “measurement”, and unitary evolution in between [29], and is therefore a natural choice of representation, with apparent physical

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<sup>5</sup>In (2.14), the dynamics is encoded in the time evolution of the Heisenberg projections. In the program of Isham *et al.*, where everything is written in terms of Schrödinger projections on  $\otimes^k H_S$  as in (2.12), the dynamics is unfolded from the histories and instead bound up into the single operator  $X$  of (2.9), which describes the boundary conditions as well. (See [3] or [6] for how to do so in the case of a canonical decoherence functional with  $\rho_\omega = 1$ ; *cf.* (2.17) below. It is not difficult to generalize the algorithm given there to more general cases, for instance, to the case  $\rho_\omega \neq 1$ .) In this work, however, where I deal with histories in  $H_S$  in the abstract, the projections will be arbitrary, and the question of dynamics need never arise.

significance, in ordinary quantum mechanics.<sup>6</sup> This fact alone makes it worthwhile to study the representation of histories by class operators independently of their representation as projection operators. For more detail, see for instance [2].

In any event, while the formulation of quantum temporal logic on  $\otimes^k H_S$  has certain mathematical advantages, it has the feature that (unless  $X$  has a very special form; see footnote 5) there is no immediate analogue other than consistency itself of the “wave function collapse”-like behaviour so characteristic of ordinary quantum mechanics. This is of no particular concern to Isham and Linden, whose aims include *generalizing* quantum theory to timeless theories, but it is not always desirable to toss out *a priori* such familiar and important physics. Considerable attention will therefore be given to decoherence functionals on the representation of histories by class operators. We thus can study the effect of retaining this “quantum branching” behaviour, while generalizing other aspects of quantum theory.

To control the proliferation of notation I will show no shame in often denoting by  $h$  an operator representative (2.12) or (2.14) of the history (2.11). (In section III C this abuse will be extended even further to include linear combinations of such representatives.) As the saying goes, which is meant should be evident from the context.

### *The History Representation Spaces $\mathcal{R}_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{C}}$*

When  $k > 1$  (which I shall always assume), there are thus two distinct possible choices for the collection of operator representatives of the fine grained histories  $(P_{\beta_1}^1, \dots, P_{\beta_{k-1}}^{k-1}, P_{\beta_k}^k)$  with temporal support  $\mathcal{T}_k$ , the projections (2.12) or the class operators (2.14). I shall call the former choice the (faithful) *projection representation*  $\mathcal{R}_{\mathcal{P}}(\mathcal{U})$  of the fine grained histories in  $\mathcal{U}$ , the latter the (unfaithful) *class operator representation*, or sometimes, the *conventional representation*,  $\mathcal{R}_{\mathcal{C}}(\mathcal{U})$ . (In either case,  $\neg h$  is represented by  $1 - h$ ,  $h \oplus h'$  by  $h + h'$ , and so on. Warning: The definitions of  $\mathcal{R}_{\mathcal{P}}$  and  $\mathcal{R}_{\mathcal{C}}$  will be extended shortly, after consideration has been given to the allowed coarse grainings.) This distinction is useful because while it may be most proper for the purposes of rigorous analysis or quantum logic to employ  $\mathcal{R}_{\mathcal{P}}$ , it is in fact the case that it is as the class operators (2.14) that histories which arise in actual physical problems enter. (This is more or less true even for the decoherence functionals which have been suggested for spacetimes which contain closed timelike curves [41,42]. The “more or less” refers to the way in which a nonunitary time evolution enters into the problem. This does not however change the fact that histories end up being represented by a collection of operators in  $\mathcal{L}(H_S)$  with all essential properties the same as  $\mathcal{R}_{\mathcal{C}}$ ; cf. the beginning of section III C.) Put another way, the decoherence functionals that have been discussed in the literature so far are sensitive only to the information about histories in  $\mathcal{U}$  contained in  $\mathcal{R}_{\mathcal{C}}$ . While the roots of this in ordinary quantum mechanics (as described above) are clear, I think it is safe to say that the meaning of this fact is not fully understood.

(For this reason, much of the work that follows is done with  $\mathcal{R}_{\mathcal{C}}$  foremost in mind. This

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<sup>6</sup>Notice that no notion of measurement is required, however, it may be recovered in appropriate circumstances [1,2,35]. Nevertheless, other potentially serious ambiguities arise. For discussion, see [21,36–40].

applies particularly to sections V and VII. Nonetheless, the general structural results of the present investigation are applicable to either method of representing the histories  $h$ , though a few are more immediately *useful* for decoherence functionals on class operators. I will try and make a note of it whenever a result has a usefulness more specific to one representation than the other.)

### *Homogeneous and Inhomogeneous Coarse Grainings: $\overline{\mathcal{R}}_{\mathcal{P}}$ and $\overline{\mathcal{R}}_{\mathcal{C}}$*

Next, the permissible coarse grainings must be specified. Recall that a coarse graining is a partition of an exhaustive and exclusive collection of fine grained histories in  $\mathcal{U}$  into exclusive classes. What is the meaning of this in  $\mathcal{R}_{\mathcal{P}}$  and  $\mathcal{R}_{\mathcal{C}}$ ? Partitions of histories disjoint in  $\mathcal{U}$  clearly correspond to operator sums in either representation. Such operator sums divide themselves into two distinct classes. In either representation, sums of projections *at one time* still have the form (2.12) or (2.14), except the projections are no longer required to be one dimensional. Following Isham, any such histories are called “homogeneous”. More generally, sums of homogeneous histories are *not* in general again of the form (2.12) or (2.14), but are sums of such terms. Such histories are called “inhomogeneous”. (Neither homogeneous nor inhomogeneous class operators are projection operators in general, but sums of *disjoint* projections (2.12) in  $\mathcal{R}_{\mathcal{P}}$  are always again projection operators.) There are therefore two obvious choices for the classes of permissible coarse grainings: “homogeneous” coarse grainings, where the allowed partitions are those for which the operator representatives  $\bar{h}$  of coarse grained histories are all required to be again homogeneous; or more generally, *arbitrary* partitions of exclusive, exhaustive sets might be allowed, corresponding to admitting arbitrary sums of operators in exclusive, exhaustive sets  $\mathcal{S} \subset \mathcal{R}_{\mathcal{P}}$  or  $\mathcal{S} \subset \mathcal{R}_{\mathcal{C}}$  (“inhomogeneous” coarse grainings, naturally.)

From the point of view of generalized quantum theory, there seems no obvious physical reason to exclude inhomogeneous coarse grainings. Indeed, through the course of this investigation it will become apparent that from the point of view of the quantum mechanics of history, it appears unnatural to restrict to only the homogeneous coarse grainings. However, as emphasized by Hartle in [2, section IV.2], restricting to homogeneous coarse grainings ensures that we can maintain in ordinary quantum mechanics a description of the system  $S$  by a single state vector evolving unitarily in time between “wave function reductions”. Admitting inhomogeneous coarse grainings is therefore to be regarded as an extension of ordinary Hamiltonian quantum mechanics. Moreover, as will become evident, admitting inhomogeneous coarse grainings has stronger mathematical consequences. I will therefore not gloss over the distinction between homogeneous and inhomogeneous coarse grainings.

In order to avoid wanton proliferation of notation,<sup>7</sup> the definition of the history representation spaces  $\mathcal{R}_{\mathcal{P}}$  and  $\mathcal{R}_{\mathcal{C}}$  is extended to include not only the fine grained histories (2.12) and (2.14), but also all the histories that may be obtained from them by homogeneous coarse grainings.  $\overline{\mathcal{R}}_{\mathcal{P}}$  and  $\overline{\mathcal{R}}_{\mathcal{C}}$  will be used to denote the even larger operator spaces obtained by including the inhomogeneously coarse grained histories as well. These spaces are then

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<sup>7</sup>Appendix B summarizes the important notation for ease of reference.

Hilbert space realizations of the abstract space of all histories  $\mathcal{U}$ .

It should be reasonably obvious that

$$\mathcal{R}_{\mathcal{P}}(\otimes^k H_S) \subset \overline{\mathcal{R}}_{\mathcal{P}}(\otimes^k H_S) \subset \mathcal{P}(\otimes^k H_S) \subset \mathcal{L}(\otimes^k H_S) \quad (2.15)$$

and that

$$\mathcal{P}(H_S) \subset \mathcal{R}_{\mathcal{C}}(H_S) \subset \overline{\mathcal{R}}_{\mathcal{C}}(H_S) \subset \mathcal{L}(H_S), \quad (2.16)$$

where  $\mathcal{P}(H)$  is the collection (lattice) of projections on  $H$ . All of these inclusions are proper so long as  $k > 1$ .

Note that  $\overline{\mathcal{R}}_{\mathcal{P}}(\otimes^k H_S) \neq \mathcal{P}(\otimes^k H_S)$  because  $\overline{\mathcal{R}}_{\mathcal{P}}$  includes only sums of *disjoint* projection operators, and it is *not* true that all projection operators on  $\otimes^k H_S$  can be written as sums of disjoint homogeneous projections [4]; those which cannot are termed “exotic”. Their physical interpretation remains at best unclear. (Isham, Linden, and Schreckenberg [6] understandably circumvent this irritation by fiat, choosing to work with  $\mathcal{P}(\otimes^k H_S)$  exclusively.)

The meaning of the inclusions (2.15) and (2.16) is that  $\mathcal{R}_{\mathcal{C}}$  is “larger” than  $\mathcal{R}_{\mathcal{P}}$ , in the sense that  $\mathcal{R}_{\mathcal{C}}$  “fills up” more of the space of operators  $\mathcal{L}$  in which it resides than does  $\mathcal{R}_{\mathcal{P}}$ . The most striking manifestation of this observation is the “ray-completeness” property of  $\overline{\mathcal{R}}_{\mathcal{C}}(H_S)$  in  $\mathcal{L}(H_S)$  that I discuss just below. This property actually implies that a decoherence functional on  $\overline{\mathcal{R}}_{\mathcal{C}}$  is a positive functional, as will be seen in section III C. Taking the representation space to be  $\overline{\mathcal{R}}_{\mathcal{C}}$  therefore has stronger mathematical consequences than taking the representation space to be anything else.

To avoid needless repetition,  $\mathcal{R}$  will frequently stand in for either of  $\mathcal{R}_{\mathcal{P}}$  or  $\mathcal{R}_{\mathcal{C}}$ , and  $\overline{\mathcal{R}}$  similarly for  $\overline{\mathcal{R}}_{\mathcal{P}}$  or  $\overline{\mathcal{R}}_{\mathcal{C}}$ . (I will not always bother to say “ $\mathcal{R}$  or  $\overline{\mathcal{R}}$ ” when it is reasonably clear that either will do.)  $H$  will similarly stand for either  $H_S$  or  $\otimes^k H_S$ , as appropriate. The reason it is useful to consolidate the discussion of decoherence functionals on these representation spaces is that most of the structural results for decoherence functionals discussed in sections III and beyond depend in no way on the details of the chosen representation space  $\mathcal{R}$ . It is enough that all of these contain a basis for the operator space in which they are contained, as I now discuss.

#### *How Much of $\mathcal{L}(H)$ Does $\mathcal{R}(H)$ Fill? The Ray-Completeness of $\overline{\mathcal{R}}_{\mathcal{C}}$*

There are two points that call for emphasis. First, by definition it is only sums of *disjoint* histories which are defined in  $\mathcal{R}$  or  $\overline{\mathcal{R}}$  by coarse graining. The second, related, point is that the  $\mathcal{R}$ ’s are *not linear spaces*; linear combinations of histories in  $\mathcal{R}$  are not usually again histories. Nevertheless, in spite of not being linear spaces,  $\mathcal{R}_{\mathcal{P}}(\otimes^k H_S)$  contains a basis for  $\mathcal{L}(\otimes^k H_S)$ , and  $\mathcal{R}_{\mathcal{C}}(H_S)$  likewise contains a basis of  $\mathcal{L}(H_S)$ . In other words, *the linear, or vector space, completion of any of the spaces  $\mathcal{R}(H)$  is the full space of operators  $\mathcal{L}(H)$ .*

This is true whatever the admitted coarse grainings; under our democratic assumptions a basis is already contained in the full collection of fine grained histories (2.12) or (2.14). Moreover, so long as  $k > 3$  (which I shall henceforth always assume), and inhomogeneous coarse grainings are permitted,  $\overline{\mathcal{R}}_{\mathcal{C}}$  contains a vector (operator) along every ray in  $\mathcal{L}(H_S)$ . This is the “ray-completeness” property of  $\overline{\mathcal{R}}_{\mathcal{C}}(H_S)$  in  $\mathcal{L}(H_S)$ . (This statement is not meant



to be obvious. A proof appears in appendix A.) In this sense,  $\overline{\mathcal{R}}_{\mathcal{C}}(H)$  contains, up to normalization, *every* basis for  $\mathcal{L}(H_S)$ . The ray-completeness property is *not* true of  $\overline{\mathcal{R}}_{\mathcal{P}}$ , or even  $\mathcal{R}_{\mathcal{C}}$ , *i.e.* if only homogeneous coarse grainings are allowed.

The fact that the linear completion of each representation space  $\mathcal{R}$  is the full space of operators  $\mathcal{L}$  will become the basis for extending the domain of definition of  $d$  to all of  $\mathcal{L}$  in the next section. The ray-completeness of  $\overline{\mathcal{R}}_{\mathcal{C}}$  is strong enough to imply that a decoherence functional defined on  $\overline{\mathcal{R}}_{\mathcal{C}}$  is always a positive functional when extended to  $\mathcal{L}$ . (That is not true for any of the other  $\mathcal{R}$ 's.)

The rest of this section, and all of the next (section IID), should probably be skipped on a first reading.

To close this subsection, let me emphasize the fact that the  $\mathcal{R}$ 's are not linear spaces by relaying some observations about the structure of  $\mathcal{R}_{\mathcal{P}}$  and  $\mathcal{R}_{\mathcal{C}}$  that will be of some importance as we go along. First, for finite temporal supports (finite  $k$ ), being all projection operators,  $\overline{\mathcal{R}}_{\mathcal{P}}(\otimes^k H_S) \subset \mathcal{B}_1(\mathcal{L}(\otimes^k H_S))$ . That is,  $\overline{\mathcal{R}}_{\mathcal{P}}$  is contained in the unit ball in the space of linear operators on  $\otimes^k H_S$  (in the usual norm, or uniform operator, topology. The norm here is the standard operator norm on a Hilbert space  $H$ ,  $\|\mathcal{O}\| = \sup_{\|x\|_H=1} \|\mathcal{O}x\|_H$ .) Similarly, as each homogeneous class operator (2.14) is merely a product of projection operators, the norm of each such homogeneous class operator is bounded by 1,  $\|C_{\beta}\| \leq 1$ , so that  $\mathcal{R}_{\mathcal{C}} \subset \mathcal{B}_1(\mathcal{L}(H_S))$ . However,  $\overline{\mathcal{R}}_{\mathcal{C}}$  can be considerably larger, as inhomogeneous coarse grainings permit histories which are sums of many class operators which point in almost the same “direction”. An upper bound in finite dimensions comes from the observation that a coarse grained history is a sum of fine grained ones, and with  $\dim H_S = N$ , there are at most  $N^k$  such fine grained histories. Therefore, the triangle inequality tells us that  $\overline{\mathcal{R}}_{\mathcal{C}}(H_S) \subset \mathcal{B}_{N^k}(\mathcal{L}(H_S))$ . (One might wonder if the fact that each  $h \in \overline{\mathcal{R}}_{\mathcal{C}}$  appears in some “partition” of unity,  $1 = \sum h$ , can strengthen this bound much. At least in some specific cases, it can. Suppose  $1 = h + \overline{h}$ , supposing further that  $h$  ( $\overline{h}$ ) is a sum of  $n$  ( $\overline{n}$ ) fine grained histories, where of course  $n + \overline{n} = N$ , so that  $\|h\| \leq n$  ( $\|\overline{h}\| \leq \overline{n}$ ). Then  $\|h\| = \|1 - \overline{h}\| \leq 1 + \|\overline{h}\| \leq 1 + \overline{n}$ . This doesn't seem to be of much help in general. An exception is cases where  $\overline{h}$  is homogeneous, so that  $\|\overline{h}\| \leq 1$  (despite  $\overline{n} > 1$ ) and therefore  $\|h\| \leq 2$ .)

#### D. Reconstruction of History Sequences from History Operators<sup>8</sup>

It was remarked in the paragraph following (2.14) that given only two homogeneous class operators, there is no way in general to determine whether or not they represent disjoint physical histories (2.11). Continuing this discussion, it is interesting to enquire, given an operator in  $\mathcal{R}_{\mathcal{P}}$  or  $\mathcal{R}_{\mathcal{C}}$ , to what history (2.11) does it correspond? A closely related problem, which will be of some importance later on, is, how does one determine whether an operator  $h \in \mathcal{L}(H_S)$  ( $h \in \mathcal{L}(\otimes^k H_S)$ ) is actually a member of  $\mathcal{R}_{\mathcal{C}}$  ( $\mathcal{R}_{\mathcal{P}}$ )?

Unfortunately, at present there do not appear to be easy answers to these questions.

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<sup>8</sup>This section should be skipped on a first reading. It is placed here for logical coherence.

### Physical History Operators and $\mathcal{R}_{\mathcal{P}}$

First consider  $\mathcal{R}_{\mathcal{P}}$ . It is of course true that every element  $h$  of  $\mathcal{R}_{\mathcal{P}}$  is a projection operator,  $h^\dagger h = h$ . However, as noted above, not every projection operator on  $\otimes^k H_S$  can be written as a sum of disjoint projections of the form (2.12). Those projections which cannot have been termed “exotic” histories by Isham; their physical interpretation, if any, remains obscure. (This is one reason to remember, at least, that  $\overline{\mathcal{R}}_{\mathcal{P}}(\otimes^k H_S)$  is actually only a proper subset of  $\mathcal{P}(\otimes^k H_S)$ .) An extension of the methods of section VII should make it possible to determine algebraically whether or not a history is a simple product of projections as in (2.12), that is, is a member of  $\mathcal{R}_{\mathcal{P}}$ . Moreover, as in section VII, it should be possible to construct the individual projections explicitly. However, there is no obvious technique for determining whether or not more general projection operators can be written as sums of *disjoint* projections, that is, are actually members of  $\overline{\mathcal{R}}_{\mathcal{P}}$ .

### Physical History Operators and $\mathcal{R}_{\mathcal{C}}$

The situation is considerably worse for  $\mathcal{R}_{\mathcal{C}}$ . As noted (and seen explicitly in appendix A), there are in general *many* histories (2.11) which might give rise to a given class operator. Further, at present there are no techniques for constructing this family of histories given only the class operator. Even worse, unless a class operator is actually just a projection, there does not appear to be any easy (practical) way to tell in general whether or not an arbitrary operator in  $\mathcal{L}(H_S)$  is actually a class operator at all, *i.e.* is a member of  $\mathcal{R}_{\mathcal{C}}$  or  $\overline{\mathcal{R}}_{\mathcal{C}}$ . This will become an issue in section V after I have shown how to extend the domain of definition of a general decoherence functional from  $\mathcal{R}(H)$  to  $\mathcal{H} = \mathcal{L}(H)$ . There I show how to solve the problem of determining what are all the “histories” (operators in  $\mathcal{H}$ ) which are consistent according to a given decoherence functional. In  $\mathcal{R}_{\mathcal{C}}$  or  $\overline{\mathcal{R}}_{\mathcal{C}}$  it remains to determine which of these operators actually correspond to physical histories.

### Bounds on the Norm of Physical History Operators

The best that can be made of the situation is the following.

By assumption (2.2),  $d(h, h) \geq 0$  if  $h \in \mathcal{R}$ . Therefore, if some operator  $n$  is found for which  $d(n, n) < 0$ , it obviously does not correspond to any physical history.<sup>9</sup> Similarly, suppose  $h \in \mathcal{D}_d$ , *i.e.*  $h$  appears in at least one set of histories consistent according to  $d$ . (In section V I show that  $\mathcal{D}_d = \{h \in \mathcal{R} \mid d(h, 1 - h) = 0\}$ .) Then (2.2) and (2.4) are sufficient to imply that  $d(h, h) \leq 1$ . (That  $0 \leq d(h, h) \leq 1$  for any consistent  $h \in \mathcal{R}$  is, of course, much of the motivation for the precise form of the definition of the decoherence functional.) Thus, any operator  $n$  for which  $d(n, n) > 1$  is not an element of  $\mathcal{D}_d$ ; though it is possible for *inconsistent*  $h \in \mathcal{R}$  to have  $d(h, h) > 1$ , precisely for this reason such histories are not usually of interest.

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<sup>9</sup>In passing I note that it is true that decoherence functionals are always positive on the *span* of each of their (physical) consistent sets of histories in  $\mathcal{H}$ ; see section III C.

On the other hand,  $0 \leq d(h, h) \leq 1$  is not (of course) sufficient to imply that  $h \in \mathcal{R}$ . Indeed,  $0 \leq d(h, h) \leq 1$  for *any* “consistent” operator  $h \in \mathcal{H} = \mathcal{L}(H)$  if  $d$  is positive on  $\mathcal{H}$ . This includes in particular the important case of the canonical decoherence functional.

(Given the definition of  $\mathcal{D}_d$  noted in the previous paragraph, it will come as no surprise that the usage of the term “consistent” here merely means that  $h \in \mathcal{O}_d$ , where  $\mathcal{O}_d$  is defined in the same way as  $\mathcal{D}_d$ , namely,  $\mathcal{O}_d = \{h' \in \mathcal{H} \mid d(h', 1 - h') = 0\}$ . In other words,  $\mathcal{O}_d$  is the generalization of  $\mathcal{D}_d$  to *any* operator in  $\mathcal{H}$ , not just those that correspond to physical histories. If  $d$  is positive and  $h \in \mathcal{O}_d$ ,  $d(h, h) \leq 1$  for the same reason that is true if  $h \in \mathcal{D}_d$ . Similarly, it is useful to define  $\mathcal{Q}_d$ , an operator analogue of  $\mathcal{C}_d$ , as the collection of exhaustive ( $\sum h = 1_H$ ), mutually consistent sets of operators. Clearly,  $\mathcal{D}_d = \mathcal{O}_d \cap \mathcal{R}$  and  $\mathcal{C}_d = \mathcal{Q}_d \cap \{\mathcal{S}_d\}$ .)

Thus, if only homogeneous class operators are permissible ( $\mathcal{R} = \mathcal{R}_C$ ), the “probability” of an operator  $p(h) = d(h, h)$  is insufficient to tell us whether or not  $h \in \mathcal{R}_C$ .

In  $\overline{\mathcal{R}}_C$ , the situation is slightly better. To see why, remember the ray-completeness of  $\overline{\mathcal{R}}_C$ . In fact, as shown in appendix A,  $\mathcal{B}_{\frac{1}{8}}(\mathcal{L}(H_S)) \subset \overline{\mathcal{R}}_C(H_S)$ , so that if  $\|h\| \leq \frac{1}{8}$  we know that it is *some* class operator, even if we can’t reconstruct all of its associated histories (2.11).<sup>10</sup> As noted in the first appendix, it is possible to enlarge this ball as  $k$  or  $N = \dim H_S$  increases. Recall on the other hand that  $\overline{\mathcal{R}}_C(H_S) \subseteq \mathcal{B}_{N^k}(\mathcal{L}(H_S))$  (see the end of the previous section), so that the ball containing  $\overline{\mathcal{R}}_C(H_S)$  increases with  $k$  and  $N$  as well. Thus, the operator norm of  $h$  is not in general of much use in fixing whether or not  $h \in \overline{\mathcal{R}}_C$ .

### Why it Matters

The difficulty here is analogous to the inability in conventional applications of quantum theory to decide to what physical quantity an arbitrary self-adjoint operator corresponds. However, the inability to reconstruct histories from class operators is a somewhat more severe complication. For, suppose all of the “histories” (operators) consistent according to a decoherence functional over  $\mathcal{R}_C(H_S)$  – that is,  $\mathcal{O}_d$  – have been found. (In section V I show how to do this.) What good is this information if we cannot determine to what *physical* histories these operators correspond? Formally, of course, to find  $\mathcal{D}_d$  we can just intersect  $\mathcal{O}_d$  with  $\mathcal{R}_C(H_S)$ , that is,  $\mathcal{D}_d = \mathcal{O}_d \cap \mathcal{R}$ , but that is not very comforting to a physicist. At present, we can only make much *practical* use of the converse problem of determining whether a given set physical histories is consistent.

Thus, there are at least two important open problems. First, is there a practical test for determining whether or not  $h \in \mathcal{R}$ ? This is non-trivial so long as  $\mathcal{R}(H) \neq \mathcal{P}(H)$ . Second, given a class operator (homogeneous or inhomogeneous), how does one construct the family of histories which it represents? It remains a project for the future to see if this situation

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<sup>10</sup>Now, one might wonder whether or not it is possible to bound  $\|h\|$  by  $p(h)$  in some fashion. Unfortunately, this does not appear to be the case. Instead, one can demonstrate inequalities like  $\|h\|_d \leq C_d \|h\|$  and  $\|h\|_d \leq C_{d2} \|h\|_2$ , where  $\|h\|_d = \sqrt{d(h, h)}$ ,  $\|h\|_2^2 = \text{tr}_H h^\dagger h$ , and the  $C_d$ ’s are constants depending only on  $d$ . Working with a general decoherence functional in the form (3.10), these are simple consequences of standard trace-norm inequalities that can be found in, for instance, [43, chapter VI].

can be improved; one intriguing possibility is that the operators in  $\mathcal{O}_d$  not in  $\mathcal{R}_C$  may have some physical interpretation after all.<sup>11</sup> I will not permit such obstacles to deter me from analyzing other algebraic properties of decoherence functionals and their consistent sets. Nor do they otherwise mitigate the usefulness of the methods and results introduced here.

### E. The Canonical Decoherence Functional

Finally, to end this section, it is time to exhibit the “canonical” decoherence functional of ordinary quantum mechanics, in which boundary conditions are imposed at some initial and final times. Define  $d_{\alpha\omega} : \overline{\mathcal{R}}_C \times \overline{\mathcal{R}}_C \rightarrow \mathbb{C}$  by

$$d_{\alpha\omega}(h, h') = \text{tr}[\rho_\omega h^\dagger \rho_\alpha h'], \quad (2.17)$$

where  $\rho_\alpha$  and  $\rho_\omega$  are positive Hermitian operators on  $H_S$  whose product is required to be trace-class, and normalized,  $\text{tr}[\rho_\omega \rho_\alpha] = 1$ . (Out of laziness it is hard not to refer to the initial and final boundary conditions  $\rho_\alpha$  and  $\rho_\omega$  as “density operators”, even though they may not *individually* be trace class, *e.g.* in the familiar case where  $\rho_\omega = 1$  on an infinite dimensional space. Also, it is of course possible to consider restricting the domain of  $d_{\alpha\omega}$  to  $\mathcal{R}_C \times \mathcal{R}_C$ , however, there appears to be no particular profit in this constraint unless  $\rho_\omega = 1$ , in which case the restriction to  $\mathcal{R}_C$  guarantees that the familiar notion of “quantum state at a moment of time” may be defined [2, section IV.2].)

The origin of the definition (2.17) lies in the old [44,45] formula for the probability of measuring the eigenvalue  $a$  of observable  $A$ ,  $p(a) = \text{tr}[\rho P_a]$ , where  $\rho$  is the initial density matrix. Using the fact that projections satisfy  $P_a P_{a'} = \delta_{aa'}$ , writing it out it is not difficult to see that  $\text{tr}[C_\beta^\dagger \rho C_\beta]$  is the joint probability for the alternatives in the history represented by  $C_\beta$  (*cf.* (2.14)) to be realized upon a sequence of measurements. Some thirty years ago, [46] extended this formula to the case where there is also a nontrivial final boundary condition  $\rho_\omega$ . Finally, Gell-Mann and Hartle [10] generalized the probability formula to measure the interference between histories, as well as determine their probabilities.

For culture, I note that it is possible by the standard methods to rewrite (2.17) (or, indeed, any decoherence functional on  $\overline{\mathcal{R}}_C$ ) as a functional integral, but that is not a topic that will be explored here.

In spite of the rather general tone of this work, the canonical decoherence functional is obviously a most important case to consider, not least because (with  $\rho_\omega = 1$ ) it is the decoherence functional of ordinary quantum mechanics. It has several nice properties, among the most useful of which is that it is positive on all of  $\mathcal{H} = \mathcal{L}(\mathcal{H})$  (see section III F), which in turn implies other useful properties such as the Cauchy-Schwarz inequality (3.4).

## III. THE DECOHERENCE FUNCTIONAL AS HERMITIAN FORM

This portion of the paper is devoted to a discussion of the decoherence functional as an operator on histories. Section III A shows how a decoherence functional naturally defines

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<sup>11</sup>I would like to thank J.B. Hartle for emphasizing this possibility to me.

an Hermitian form on the full space of operators in which the histories live. Sections IIIB and IIIC discuss some immediate consequences of this observation, a bound on the maximum number of histories in a consistent set, and a general Cauchy-Schwarz inequality for positive decoherence functionals. Section IIID makes use of the point of view that  $d$  is an operator on histories by introducing some familiar and useful notation in the context of the quantum mechanics of history, and develops some powerful tools for calculating with a general decoherence functional. Section IIIE shows how the ILS theorem classifying decoherence functionals pops out almost magically from this formalism, and section IIIF exhibits the canonical decoherence functional as an example illustrative of the preceding sections.

### A. Extending the Domain of $d$ from $\mathcal{R}(H)$ to $\mathcal{H} = \mathcal{L}(H)$

In this section I show how – relying on a result of Wright [8] – to extend straightforwardly the domain of definition of a decoherence functional  $d$  to all of the operators on the pertinent Hilbert space ( $H_S$  or  $\otimes^k H_S$ , in the preceding section.)  $d$  is then an inner-product like form, and the positive  $d$ 's (like the ones arising in ordinary quantum mechanics) actually *are* semi-inner products on  $\mathcal{L}$ . As evidence of the utility of this geometric perspective, some consequences of this identification are discussed in the subsequent subsections. These include a Cauchy-Schwarz inequality for positive decoherence functionals, a bound on the maximum number of histories in any consistent set (therefore implying in general a fact which has long been observed in practice, that some coarse graining is inevitably required for decoherence), and more generally, a very useful geometric point of view on the nature of consistent sets of histories and the decoherence functionals which define them.

The properties of decoherence functionals studied in the sequel do not generally depend on the specific structure of the space of operators chosen to represent quantum histories, but rather depend only on the algebraic properties of the decoherence functional embodied in (2.1-2.4). The only property of the history representation spaces that is truly essential is that their linear completion is the entire space of operators in which they are contained.<sup>12</sup>

(Actually, as we shall see in a moment, rigor currently requires that  $\mathcal{P}(H) \subseteq \mathcal{R}(H)$  for the results of this paper to be rigorously guaranteed to apply to any  $d$  on  $\mathcal{R}$ .)

Let me therefore condense and unify the notation and treatment a bit. To be specific, I am going to consider decoherence functionals  $d : \mathcal{R}_{\mathcal{P}} \times \mathcal{R}_{\mathcal{P}} \rightarrow \mathbb{C}$  or  $d : \mathcal{R}_{\mathcal{C}} \times \mathcal{R}_{\mathcal{C}} \rightarrow \mathbb{C}$  (or  $\overline{\mathcal{R}}_{\mathcal{P}} \times \overline{\mathcal{R}}_{\mathcal{P}}$  or  $\overline{\mathcal{R}}_{\mathcal{C}} \times \overline{\mathcal{R}}_{\mathcal{C}}$ ). Simplify, as before, by letting  $\mathcal{R}$  stand for  $\mathcal{R}_{\mathcal{P}}$  or  $\mathcal{R}_{\mathcal{C}}$ , and  $\overline{\mathcal{R}}$  for  $\overline{\mathcal{R}}_{\mathcal{P}}$  or  $\overline{\mathcal{R}}_{\mathcal{C}}$ . Similarly,  $H$  will stand for either of  $H_S$  or  $\otimes^k H_S$ , as appropriate.<sup>13</sup> The discussion

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<sup>12</sup>An exception to the rule that the specific choice of  $\mathcal{R}$  does not matter very much concerns the positivity of  $d$ . As emphasized in section IIIC, the ray-completeness property of  $\overline{\mathcal{R}}_{\mathcal{C}}$  (*cf.* the end of section IIC) is strong enough to imply that a decoherence functional on  $\overline{\mathcal{R}}_{\mathcal{C}}$  is actually positive on all of  $\mathcal{L}(H_S)$ . The equivalent statement is *not* true for any of the other  $\mathcal{R}$ 's, which as noted in (2.15-2.16) do not “fill up” as much of  $\mathcal{L}(H)$  as does  $\overline{\mathcal{R}}_{\mathcal{C}}(H)$ .

<sup>13</sup>As mentioned in section IIA, no *a priori* notion of time is required for the definition of a

will therefore be about a general decoherence functional  $d : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{C}$  (or  $d : \overline{\mathcal{R}} \times \overline{\mathcal{R}} \rightarrow \mathbb{C}$ ) whose histories are operators on some Hilbert space  $H$ .

With this simplification in hand, let me proceed.

The goal is to extend a decoherence functional  $d$  defined on  $\mathcal{R}$  (or  $\overline{\mathcal{R}}$ )<sup>14</sup> to a sesquilinear form defined on all of  $\mathcal{H} = \mathcal{L}(H)$ . Because the linear completion of any of the spaces  $\mathcal{R}(H)$  is  $\mathcal{L}(H)$ , this is, on the face of it, straightforward. We are given an Hermitian, positive, normalized function  $d : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{C}$  which is bi-additive on disjoint histories. Leaving aside positivity and normalization for the moment, begin by defining a bilinear functional  $b : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  by

$$b(h, \alpha h' + \beta h'') = \alpha d(h^\dagger, h') + \beta d(h^\dagger, h'') \quad (3.1)$$

for all complex numbers  $\alpha$  and  $\beta$  and for all histories  $h, h', h'' \in \mathcal{R}$  (whether or not they are disjoint; the sum on the left hand side is well defined in  $\mathcal{H}$  irrespective, and this definition thereby extends additivity of  $d$  to *all* pairs of histories), and similarly

$$b(\alpha h' + \beta h'', h) = \alpha d(h'^\dagger, h) + \beta d(h''^\dagger, h). \quad (3.2)$$

By obvious abuses of notation, finally define  $d : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  by

$$d(h, h') = b(h^\dagger, h'). \quad (3.3)$$

The  $d$  defined by (3.3) is then an Hermitian, normalized, sesquilinear form on  $\mathcal{H}$  that equals  $d$  when restricted to  $\mathcal{R} \times \mathcal{R}$ .

Recall that all the  $\mathcal{R}$ 's contain at least one basis of  $\mathcal{H}$ , so the extension is defined on all of  $\mathcal{R}$ . Moreover, as finite linear combinations of vectors in  $\mathcal{R}$  generate  $\mathcal{H}$  in finite dimensions – the collection of finite linear combinations is actually dense in  $\mathcal{H}$  in infinite dimensions – this extension is essentially unique. In order to guarantee that the extension is consistently defined, we must appeal to a result of Wright [8], which amounts even in infinite dimensions to a rigorous demonstration that a bounded decoherence functional on  $\mathcal{P}(H)$  can be linearly extended in this way.<sup>15</sup>

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generalized quantum theory. In the general case, therefore, there will be no immediate analog of the representation spaces  $\mathcal{R}_{\mathcal{P}}$  or  $\mathcal{R}_{\mathcal{C}}$  that are present in theories with a time. However, though there of course are no general rules for what collections of operators can represent the “histories” of an arbitrary generalized quantum theory, from the point of view of quantum logic it would not be unusual to assume that there is a representation in terms of something like the lattice of projections on a Hilbert space. In any case, as just argued, very little of what follows depends on the details of the choice of  $H$  – for most purposes, all that really matters is that the linear completion of  $\mathcal{R}(H)$  is  $\mathcal{L}(H)$  – so that most of the discussion should be applicable to any Hilbert space quantum theory.

<sup>14</sup>An addendum I shall from here on often not bother to repeat.

<sup>15</sup>It is important not to underestimate the importance of this highly non-trivial result to the present work; it is the key to the general applicability of the results obtained here to decoherence functionals other than the (manifestly sesquilinear) canonical one. Many of these ideas were developed before the paper by Wright appeared, but it is only his results that guarantee that the extension of  $d$  is well defined.

We must be somewhat careful here. Wright guarantees for us that the extension described above is consistent for decoherence functionals defined on  $\mathcal{P}(H)$ . However,  $\mathcal{R}_{\mathcal{P}}$  and  $\overline{\mathcal{R}}_{\mathcal{P}}$  are proper subsets of  $\mathcal{P}(H)$ ; it is not obvious (at least to me) whether or not the same guarantee holds as well for decoherence functionals defined *a priori* only on  $\mathcal{R}_{\mathcal{P}}$  or  $\overline{\mathcal{R}}_{\mathcal{P}}$ . I shall assume that it does. (This is one reason ILS are willing to tolerate the presence in  $\mathcal{P}(H)$  of the “exotic” projections mentioned below (2.16). Failure of (sesqui-)linear extendibility on  $\mathcal{R}_{\mathcal{P}}$  or  $\overline{\mathcal{R}}_{\mathcal{P}}$  would provide even stronger motivation to find a physical interpretation for the “exotic” projections in  $\mathcal{P}(H)$ .)

On the other hand,  $\overline{\mathcal{R}}_{\mathcal{C}}(H) \supset \mathcal{R}_{\mathcal{C}}(H) \supset \mathcal{P}(H)$ . In this sense, the decoherence functionals on  $\overline{\mathcal{R}}_{\mathcal{C}}(H)$  are a *subset* of the decoherence functionals on  $\mathcal{R}_{\mathcal{C}}(H)$ , which are in turn a subset of the decoherence functionals on  $\mathcal{P}(H)$ . Wright showed that every decoherence functional on  $\mathcal{P}(H)$  may consistently be (sesqui-)linearly extended; therefore every decoherence functional on  $\mathcal{R}_{\mathcal{C}}$  and  $\overline{\mathcal{R}}_{\mathcal{C}}$  may be as well.

The extension of the domain of  $d$  to all of  $\mathcal{H}$  has the distinct virtue of making available the full arsenal of algebraic tools for studying Hermitian operators. In spite of the issues raised in section IID, I will frequently lapse into the habit of referring to any element of  $\mathcal{H}$  as a “history” or “history operator”.

## B. The Maximum Number of Histories in a Consistent Set

It is well known that decoherence in general requires some coarse graining. In this section I explain one of the mathematical origins of this phenomenon, that there is an upper bound on the number of histories with non-zero probability in any consistent set. This bound turns out to be  $\dim \mathcal{H} / \mathcal{N}_d$ , where  $\mathcal{N}_d$  is the nullspace of  $d$  in  $\mathcal{H}$ . As we shall see, this is a simple consequence of the interesting fact (demonstrated below) that histories with non-zero probability must be linearly independent of one another. For the case of a canonical decoherence functional (2.17) this bound turns out to be  $r_{\alpha} r_{\omega}$ , where  $r_{\alpha}$  and  $r_{\omega}$  are the ranks of the initial and final boundary conditions. (In the special case of the canonical decoherence functional with  $\rho_{\omega} = 1$ , the existence of such a bound was noticed previously by Diosi [47] and by Dowker and Kent [21].)

Let us see how this bound comes about.

Any decoherence functional will possess, in general, a nonempty nullspace  $\mathcal{N}_d$  defined as  $\mathcal{N}_d(\mathcal{H}) = \{o \in \mathcal{H} \mid d(\cdot, o) = 0\}$ . There is then a natural consistency-preserving equivalence relation<sup>16</sup> in  $\mathcal{H}$  associated with  $\mathcal{N}_d$ ,  $h \sim h + o$ ,  $o \in \mathcal{N}_d$ , which defines the factor space  $\mathcal{H}_d = \mathcal{H} / \mathcal{N}_d$  (which is in turn naturally isomorphic to  $\mathcal{N}_d^{\perp}$ .) This is a useful space because  $d$  is non-degenerate on  $\mathcal{H}_d$  (that is,  $d(h, h') = 0 \forall h \in \mathcal{H}_d \Rightarrow h' = 0$ ).

Now, linearly dependent histories cannot be consistent with one another unless they have zero probability, for supposing  $h' = \alpha h$ ,  $d(h, h') = \alpha d(h, h)$ .<sup>17</sup>

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<sup>16</sup>While this equivalence relation on histories under  $d$  is certainly of *mathematical* relevance, there is no apparent reason that histories equivalent under this relation must be in any sense *physically* equivalent.

<sup>17</sup>While this statement is always *true*, it only has interesting *consequences* for  $\mathcal{R}_{\mathcal{C}}$ , as all disjoint

Moreover, any history in a consistent set which is linearly dependent on the other histories in the set must have zero probability, as must all the histories which it is dependent upon. In other words, *all the histories in a linearly dependent subset of a consistent set have zero probability.* To see this, consider a consistent set  $\{h_1, \dots, h_n\} \in \mathcal{C}_d$ . Suppose  $h_1 = \sum_{i=2}^m c_i h_i$  (where  $m < n$ ). Then, for  $1 < j \leq m$ ,  $0 = d(h_j, h_1) = \sum_j c_j p_j$  by the assumption of consistency, so that all the  $h_j$  with  $c_j \neq 0$  have zero probability  $p_j$ , and  $p_1 = \sum_{i=2}^m |c_i|^2 p_i = 0$  as well.

Now consider an exclusive, exhaustive set of histories. Drop all components lying in  $\mathcal{N}_d$ , thereby reducing the set from  $\mathcal{H}$  to  $\mathcal{H}_d = \mathcal{H}/\mathcal{N}_d$ . In order to be consistent, the subset of the remaining non-zero histories which all have non-zero probability must be linearly independent, and therefore have no more than  $\dim \mathcal{H}_d = \dim \mathcal{N}_d^\perp$  members.<sup>18</sup>

Thus, the maximum number of histories with non-zero probability in a consistent set is  $\dim \mathcal{H}/\mathcal{N}_d$ .

For positive definite decoherence functionals this is of course obvious, as such decoherence functionals are genuine inner products on  $\mathcal{H}$  (*cf.* section III C). In fact, in this case it is clear that the maximum number of histories in a mutually consistent set is  $\dim \mathcal{H}$ . (A consistent set is just a set of operators orthogonal under the inner product defined by  $d$ .) The broader generality of the result is nonetheless interesting.

For the canonical decoherence functional (2.17) this upper bound on the most non-zero probability histories in a decohering set is just  $r_\alpha r_\omega$ , where  $r_\alpha$  and  $r_\omega$  are the ranks of the initial and final boundary operators.

This is easy to see. Pick two bases for  $H$  consisting in the eigenvectors of  $\rho_\alpha$  and  $\rho_\omega$ ,  $\{|\alpha\rangle, |\bar{\alpha}\rangle\}$  and  $\{|\omega\rangle, |\bar{\omega}\rangle\}$ . Here, all the  $r_\alpha$  ( $r_\omega$ )  $|\alpha\rangle$  ( $|\omega\rangle$ ) have non-zero eigenvalues, and all the  $|\bar{\alpha}\rangle$  ( $|\bar{\omega}\rangle$ ) have zero eigenvalues. Then  $\mathcal{H} = \mathcal{L}(H) = \mathcal{N}_{\alpha\omega}^\perp \oplus \mathcal{N}_{\alpha\omega}$  is spanned by  $\{|\alpha\rangle\langle\omega|\} \cup \{|\bar{\alpha}\rangle\langle\omega|, |\bar{\alpha}\rangle\langle\bar{\omega}|, |\alpha\rangle\langle\bar{\omega}|\}$ , so that the dimension of  $\mathcal{N}_{\alpha\omega}^\perp \simeq \mathcal{H}/\mathcal{N}_{\alpha\omega}$  is clearly  $r_\alpha r_\omega$ .

The physical import of the existence of such an upper bound is that, in general, decoherence requires some coarse graining, a fact which has long been observed in explicit calculations.

This is because the number of histories in a completely fine-grained exclusive, exhaustive set is in general greater than the maximum number of such histories with non-zero probability,  $\dim \mathcal{H}/\mathcal{N}$ . If  $\dim H_S = N$ , in  $\overline{\mathcal{R}}_{\mathcal{P}}$  the former number is  $\dim \mathcal{H} = N^k \geq \dim \mathcal{H}/\mathcal{N}$ . In  $\overline{\mathcal{R}}_{\mathcal{C}}$  the situation is considerably more acute. There are at most  $N^2$  linearly independent histories in  $\mathcal{H} \simeq H_S \otimes H_S^*$ , while there are  $N^k$  histories in a completely fine grained exclusive, exhaustive set. In order for a set of class operators to decohere, most of its histories must have zero probability, or there must be some coarse graining. (In general both will be true.)

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histories in  $\mathcal{R}_{\mathcal{P}}$  are linearly independent.

<sup>18</sup>It is perhaps worth mentioning that in deriving this bound we have made no use of the fact that the histories in a consistent set must sum to the unit in  $H$ . This will give rise to a consistency condition – see (5.1) – which may further reduce the number of allowed non-zero probability consistent histories for a given decoherence functional.



### C. Positivity: a Cauchy-Schwarz Inequality, and the Decoherence Functional as Inner Product

In this section, I show that positive decoherence functionals obey a Cauchy-Schwarz inequality in  $\mathcal{H}$ .

By definition,  $d$  is positive on  $\mathcal{R}$ . This is not, however, sufficient in general to imply positivity of  $d$  on all of  $\mathcal{H}$  unless  $\mathcal{R} = \overline{\mathcal{R}}_C$ . (Indeed, Isham and Linden [4] supply a counterexample in the  $\mathcal{P}(H)$  case, and I provide in appendix A a demonstration of the existence of a counterexample in  $\mathcal{R}_C$ .) A necessary and sufficient condition for positivity on  $\mathcal{H}$  is, for instance, the positive kernel condition (2.10). Another is that  $d$  possess no negative eigenvalues on  $\mathcal{H}$  (*cf.* (3.10)).

More generally, any decoherence functional on  $\overline{\mathcal{R}}_C$  is positive because of its ray-completeness property:  $\overline{\mathcal{R}}_C(H)$  contains a vector in every ray of  $\mathcal{L}(H)$ . Being positive on  $\overline{\mathcal{R}}_C$  therefore implies positivity on  $\mathcal{L}$ . This is *not* the case for  $\mathcal{R}_C$ . Decoherence functionals on  $\mathcal{R}_C$  or  $\mathcal{R}_P$  or even  $\mathcal{P}(H)$  may have negative eigenvalues (but are still of course positive on pairs of histories in  $\mathcal{R}$ .)

In any case, the important observation is that *positive decoherence functionals are bona fide semi-inner products on  $\mathcal{H}$*  [8]. (“Semi”, because a general  $d$  will as a rule possess a nontrivial nullspace  $\mathcal{N}$ .) The canonical decoherence functional (2.17), the decoherence functional of ordinary quantum mechanics, is obviously the most important example: *the decoherence functional of ordinary quantum mechanics is an inner product on class operators.*

A related consequence of the positivity of  $d$  is that all positive decoherence functionals satisfy a Cauchy-Schwarz inequality. (This was first noted in the case of the canonical decoherence functional with  $\rho_\omega = 1$  in [19].) That is,

$$|d(h, h')|^2 \leq d(h, h) d(h', h') \quad (3.4)$$

for positive decoherence functionals. This may be proved in the standard way: a positive decoherence functional satisfies  $d(h + \lambda h', h + \lambda h') \geq 0$  for any complex number  $\lambda$ . Supposing  $d(h', h') \neq 0$ , choosing  $\lambda = -d(h', h)/d(h', h')$  and using sesquilinearity and Hermiticity gives (3.4).

Because of (2.4) ( $d(1, 1) = 1$ ), that (3.4) hold for every  $h, h' \in \mathcal{H}$  is not only necessary, but also sufficient to imply the positivity of  $d$  on  $\mathcal{H}$ . Further, as the proof of the triangle inequality depends only on the Cauchy-Schwarz inequality (and not on positive *definiteness*), positive  $d$ 's obey

$$\sqrt{d(h + h', h + h')} \leq \sqrt{d(h, h)} + \sqrt{d(h', h')}. \quad (3.5)$$

Note that the fact that  $d$  has in general a non-empty nullspace  $\mathcal{N}$  does not impede the proof of either of these inequalities. In fact, from (3.4) it is now clear that for positive  $d$ ,  $d(h, h) = 0$  implies  $d(h, h') = 0$ . This is good for consistency, because it means that zero probability members of exhaustive sets of histories cannot spoil the consistency of the set. (This could, and perhaps should, be taken as a physical argument for including positivity as an additional requirement on physical decoherence functionals!)

(In fact, the property that zero probability histories cannot interfere with other histories is actually equivalent to positivity. For, suppose  $o$  is a zero probability history,  $d(o, o) = 0$ .

Consider the condition  $d(h, o) = 0 \forall h \in \mathcal{H}$ . Without loss of generality we can take  $h, o \in \mathcal{N}^\perp$ . As noted previously, decoherence functionals are non-degenerate when restricted to  $\mathcal{N}^\perp$ , so the condition implies  $o = 0$  and so  $o \in \mathcal{N}$  (trivially) after all. But non-positive decoherence functionals do have zero probability histories which do not live in  $\mathcal{N}$ , which therefore must interfere with at least some other  $h \in \mathcal{N}^\perp$ .)

Finally, it is worth noting that the implications of positivity are of more general significance for the following reason: decoherence functionals are *always* positive on the span of each of their physical consistent sets (span  $\mathcal{S}$ , where  $\mathcal{S} \in \mathcal{C}_d$ ), so that, for instance, a decoherence functional which possesses a maximally fine grained consistent set (*i.e.* a consistent set with  $\dim \mathcal{H}$  linearly independent members; see the previous section) is positive on all of  $\mathcal{H}$ . In any event, this means that (3.4) and related results hold on each and every span  $\mathcal{S}$ , regardless of whether  $d$  is positive on all of  $\mathcal{H}$ .

(It is very easy to see why decoherence functionals are positive on the span of each of their physical consistent sets, whether or not  $d$  is a positive operator: supposing that we have coarse-grained all the zero probability histories into one, each consistent set  $\mathcal{S}_d = \{h_1, \dots, h_m\}$  is linearly independent, and so its members serve as a basis for span  $\mathcal{S}_d$ . But  $\mathcal{S}_d$  is consistent, so  $d(h_i, h_j) = p_i \delta_{ij}$ . Because the  $p_i$  are non-negative for physical consistent sets,  $d$  is positive – though not necessarily positive *definite* – on span  $\mathcal{S}_d$ .)

#### *A Note on Approximate and Exact Consistency of Histories*

In passing, I note a conjecture of Dowker and Kent [21], that “close to” any approximately consistent set there is an exactly decohering one. (The physical meaning of an approximately consistent set of histories in generalized quantum theory is discussed in [1], and various mathematical aspects treated in [19–22].) For positive definite decoherence functionals at least, this conjecture can be verified in the sense that “close to” an approximately orthogonal set of vectors there is always a precisely orthogonal set. In particular, the Gram-Schmidt procedure may be applied to the approximately orthogonal set in such a way that it disturbs the set “as little as possible,” in the sense that the sum of the lengths of the Gram-Schmidt “correction” vectors is minimized [48].

It is worth observing [22] that, depending on the degree of inconsistency permitted, there may be *more histories in an approximately consistent set than are allowed in an exactly consistent one* (section III B). The conjecture will therefore always fail for approximately consistent sets which have too many members.

The complicating issue here is, as ever, the question of whether the resulting operators do, in fact, correspond to physical histories, that is, are always in  $\mathcal{R}$ . In  $\mathcal{R}_\mathcal{P}$  this is unlikely to be the case in general, and another strategy for resolving the conjecture must be applied. On the other hand, the ray-completeness of  $\overline{\mathcal{R}}_\mathcal{C}$  (section II C) makes it considerably more likely that the Gram-Schmidt tactic will yield another set of physical histories close to the original, approximately decohering, one.

Until this issue can be resolved, the physical validity of the conjecture remains in question.

## D. Notation

In section III F I exhibit the preceeding observations for  $d_{\alpha\omega}$ . In this section I introduce some calculational tools, applicable to  $d_{\alpha\omega}$  as well as in the general case, that will be of some use in what follows. The “geometric” perspective on generalized quantum theory developed here will prove to have a number of practical and intuitive advantages for performing calculations in generalized quantum theory, and is therefore worth pursuing in some detail.

Having extended a general decoherence functional to a sesquilinear Hermitian operator on the linearly extended space of histories  $\mathcal{H} = \mathcal{L}(H)$  (for some Hilbert space  $H$ ; *cf.* the remarks at the beginning of section III A), it is sometimes helpful to make use of the mathematical familiarity of this situation by employing notations adapted to it.

In finite dimensions,  $\mathcal{H} \simeq H \otimes H^*$ , which is just the statement that the linear operators on  $H$  can be built from operators of the form ( $M = (im)$ )

$$R_M \equiv R_{im} = |i\rangle\langle m| \quad (3.6)$$

for any choice of bases  $\{|i\rangle\}$  and  $\{|m\rangle\}$  for  $H$  and  $H^*$  (though of course not every basis of  $H \otimes H^*$  has this simple product form).<sup>19</sup> Now, a natural inner product on  $\mathcal{L}(H)$  is the trace, and the dual of  $\mathcal{L}(H)$  is given by the trace on  $H$ . (See, for instance, [43, section VI.6] for the status of these statements in infinite dimensional Hilbert spaces.) It is natural therefore to make the associations (mostly mere notation in finite dimensions)

$$\|A\rangle \in \mathcal{H} \Leftrightarrow A \in \mathcal{L}(H) \quad (3.7)$$

$$\langle A\| \in \mathcal{H}^* \Leftrightarrow A^\dagger \in \mathcal{L}(H)^* \simeq \mathcal{L}(H) \quad (3.8)$$

where the dual is identified via

$$\langle A\| B\rangle \equiv \text{tr}_H A^\dagger B. \quad (3.9)$$

As an Hermitian form on  $\mathcal{H}$ , a general decoherence functional may then be written always in the simple diagonal form

$$d = \sum_I w_I \|I\rangle\langle I\| \quad (3.10)$$

for some orthonormal basis  $\{\|I\rangle\} \Leftrightarrow \{E_I\}$  (where the  $E_I$  are trace-orthonormal,  $\langle I\| J\rangle = \text{tr}_H E_I^\dagger E_J = \delta_{IJ}$ ), so that  $d(h, h')$  is equal to

$$\begin{aligned} \langle h\| d\| h'\rangle &= \sum_I w_I \langle h\| I\rangle\langle I\| h'\rangle \\ &= \sum_I w_I \text{tr}_H h^\dagger E_I \text{tr}_H E_I^\dagger h'. \end{aligned} \quad (3.11)$$

---

<sup>19</sup>Here I am establishing the convention that the region of the alphabet from which the indices are drawn serves to indicate which basis is meant. In the present case,  $i - l$  and  $m - p$  refer to the two distinct bases which I have just introduced. In applications concerning canonical decoherence functionals, the  $\{|i\rangle\}$  and  $\{|m\rangle\}$  will be eigenbases of the initial and final conditions respectively, as in (3.37) and (3.38).

For later use, note that we may continue rewriting this as

$$\begin{aligned}
\langle h \| d \| h' \rangle &= \sum_I w_I \operatorname{tr}_{H \otimes H} (h^\dagger E_I \otimes h' E_I^\dagger) \\
&= \sum_I w_I \operatorname{tr}_{H \otimes H} [(h^\dagger \otimes h')(E_I \otimes E_I^\dagger)] \\
&= \operatorname{tr}_{H \otimes H} [(h^\dagger \otimes h') \underline{d}],
\end{aligned} \tag{3.12}$$

using the fact that  $\operatorname{tr}_{H \otimes H} A \otimes B = \operatorname{tr}_H A \operatorname{tr}_H B$ , and where  $\underline{d}$  is defined as in (3.17) below.

More generally, any operator  $g : \mathcal{H} \rightarrow \mathcal{H}$  can be written

$$g = \sum_{IJ} g_{IJ} \|I\rangle \langle J\| \tag{3.13}$$

so that

$$\langle h \| g \| h' \rangle = \sum_{IJ} h^{I*} h'^J g_{IJ} \tag{3.14}$$

where

$$\begin{aligned}
h^I &= \langle I \| h \rangle \\
&= \operatorname{tr}_H E_I^\dagger h.
\end{aligned} \tag{3.15}$$

Or, what's the same,

$$\begin{aligned}
\langle h \| g \| h' \rangle &= \sum_{IJ} g_{IJ} \operatorname{tr}_H h^\dagger E_I \operatorname{tr}_H E_J^\dagger h' \\
&= \sum_{IJ} g_{IJ} \operatorname{tr}_{H \otimes H} (h^\dagger \otimes h')(E_I \otimes E_J^\dagger) \\
&\equiv \operatorname{tr}_{H \otimes H} [(h^\dagger \otimes h') \underline{g}],
\end{aligned} \tag{3.16}$$

where the operator  $\underline{g}$  on  $H \otimes H$  is defined as

$$\underline{g} = \sum_{IJ} g_{IJ} E_I \otimes E_J^\dagger. \tag{3.17}$$

It is therefore often convenient to exploit the isomorphism between  $\mathcal{H} \simeq H \otimes H^*$  and  $\underline{\mathcal{H}} \equiv H \otimes H$  by making the natural association between the operator  $g : \mathcal{H} \rightarrow \mathcal{H}$  and the operator  $\underline{g} : \underline{\mathcal{H}} \rightarrow \underline{\mathcal{H}}$ .

(Though these equations have been written using a general basis  $\{E_I\} \equiv \{\|I\rangle\}$  of trace-orthonormal operators on  $H$ , factor bases like that introduced in (3.6) are often convenient in calculations. For example, with  $E_I \equiv |i\rangle \langle m|$  and  $|im\rangle \equiv |i\rangle \otimes |m\rangle$  (and, of course, in the same way  $J = (jn)$ ),  $\underline{g}$  may be written

$$\underline{g} = \sum_{ijmn} g_{im,jn} |in\rangle \langle mj|, \tag{3.18}$$

*cf.* (3.41). The single versus double bars on the kets should be adequate to distinguish states of  $H \otimes H$  from those of  $H \otimes H^*$ .)

More generally, a comparison of (3.13) and (3.17) shows that, at least in the present context, it is natural and useful to identify the operator  $\|A\rangle\langle B\|$  on  $\mathcal{H} \simeq H \otimes H^*$  and the (tensor product) operator  $A \otimes B^\dagger$  on  $\underline{\mathcal{H}} \equiv H \otimes H$ . Some might even prefer to regard the right hand side of (3.16) as defining the left, with the double-bar Dirac notation merely serving as operating instructions when the operator in (3.17) is employed, through (3.16), as a sesquilinear operator on the history space  $\mathcal{H}$ . In any case, I will eventually lapse into the habit of not bothering to distinguish between them with the underscore. Nevertheless, beware that some care is then required in interpreting, for instance, operator products computed in the Dirac notation, *cf.* (3.24) versus (3.25), and (3.29), below. The reason for this is that, in spite of the fact that  $\mathcal{H} \simeq \underline{\mathcal{H}}$ , many common operations such as taking products and traces give completely different answers depending on the space in which they are interpreted. That is, our map between operators on  $\mathcal{H}$  and on  $\underline{\mathcal{H}}$  does not naturally preserve such relations. It will therefore prove necessary to distinguish these operations notationally, which however is a small fee for a useful dual service.

In this spirit, a few more pieces of notation will prove useful in later sections. For operators on  $\underline{\mathcal{H}} = H \otimes H$  it is usual to define adjoints and operator products through

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \quad (3.19)$$

and

$$(A \otimes B)(C \otimes D) = AC \otimes BD. \quad (3.20)$$

However, in order to preserve the useful correspondence between the operators  $\underline{g}$  on  $\underline{\mathcal{H}}$  of the form (3.17), and the more comforting appearance of the operator  $g$  on  $\mathcal{H}$  introduced in (3.13), it is helpful to also have available the definitions

$$(A \otimes B)^* = B^\dagger \otimes A^\dagger \quad (3.21)$$

and

$$(A \otimes B) \odot (C \otimes D) = (\text{tr}_H BC) A \otimes D, \quad (3.22)$$

extended by linearity in the obvious way. Thus, for instance,

$$\begin{aligned} \langle h' \| g \| h \rangle^* &= \sum_{IJ} g_{IJ}^* \text{tr}_{H \otimes H} (h'^\dagger E_I \otimes E_J^\dagger h)^* \\ &= \sum_{IJ} g_{IJ}^* \text{tr}_{H \otimes H} (h^\dagger E_J \otimes E_I^\dagger h') \\ &= \sum_{IJ} g_{IJ}^* \text{tr}_H (h^\dagger E_J) \text{tr}_H (E_I^\dagger h') \\ &= \sum_{IJ} g_{IJ}^* \langle h \| J \rangle \langle I \| h' \rangle \\ &= \langle h \| g^* \| h' \rangle, \end{aligned} \quad (3.23)$$

maintaining the correspondence between  $g^*$ , by definition the true adjoint of  $g : \mathcal{H} \rightarrow \mathcal{H}$ , and the operator  $\underline{g}^*$  on  $\underline{\mathcal{H}}$ , in the way we would desire. Similarly, we of course want to write

$$\begin{aligned}
fg &= \sum_{IJ} \sum_{KL} (f_{IJ} \|I\rangle\langle J\|) (g_{KL} \|K\rangle\langle L\|) \\
&= \sum_{IL} \left( \sum_J f_{IJ} g_{JL} \right) \|I\rangle\langle L\|,
\end{aligned} \tag{3.24}$$

to which corresponds

$$\begin{aligned}
\underline{f} \odot \underline{g} &= \sum_{IJ} \sum_{KL} (f_{IJ} E_I \otimes E_J^\dagger) \odot (g_{KL} E_K \otimes E_L^\dagger) \\
&= \sum_{IJ} \sum_{KL} f_{IJ} g_{KL} (\text{tr}_H E_J^\dagger E_K) E_I \otimes E_L^\dagger \\
&= \sum_{IL} \left( \sum_J f_{IJ} g_{JL} \right) E_I \otimes E_L^\dagger.
\end{aligned} \tag{3.25}$$

The operator product  $\odot$  therefore (correctly) conveys the idea that the product in (3.24) is a composition of maps. Because of the comments below (3.18), however, I will generally write expressions like the left hand side of (3.24) as  $f \odot g$  to make it absolutely clear it is the natural operator product on  $\mathcal{H}$  that is meant, facilitating the temptation to drop the underscores from the equivalent operators on  $\underline{\mathcal{H}} = H \otimes H$ . (Section V D, in which  $\underline{\mathcal{H}}$  plays no role, is an exception.)

In addition, in accord with the fact that (3.13) is an operator on operators (*i.e.* on  $\mathcal{H}$ ), it is useful to have a meaning assigned to expressions like “ $(A \otimes B)C$ ”. If it is allowed that

$$(A \otimes B)C \equiv (\text{tr}_H BC) A \tag{3.26}$$

and

$$A(B \otimes C) \equiv (\text{tr}_H AB) C \tag{3.27}$$

(*cf.* (3.22)), then there is a natural agreement between

$$\begin{aligned}
g\|A\rangle &= \sum_{IJ} g_{IJ} \|I\rangle\langle J\| A\rangle \\
&= \sum_I \left( \sum_J g_{IJ} \text{tr}_H E_J^\dagger A \right) \|I\rangle
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\underline{g}A &= \left( \sum_{IJ} g_{IJ} E_I \otimes E_J^\dagger \right) A \\
&= \sum_I \left( \sum_J g_{IJ} \text{tr}_H E_J^\dagger A \right) E_I.
\end{aligned} \tag{3.29}$$

Further, in order that  $\langle A\|g^* \Leftrightarrow (\underline{g}A)^*$ , define

$$(\underline{g}A)^* \equiv A^\dagger \underline{g}^* \tag{3.30}$$

(and use (3.27) and (3.21).)

This technology can of course be developed further, but I have developed here the basic tools useful for most calculations.

A last observation of some use is that the normalization condition  $d(1,1) = 1$  may be worked out from, say, (3.12), to read

$$\text{tr}_{H \otimes H} \underline{d} = 1. \quad (3.31)$$

(Note from (3.12) that  $\text{tr}_{\mathcal{H}} d \neq \text{tr}_{H \otimes H} d$  because  $\sum_K E_K^\dagger \otimes E_K \neq 1 \otimes 1$ . In fact, for the canonical decoherence functional (2.17), which assumes the normalization  $\text{tr}_H \rho_\alpha \rho_\omega = 1$ ,  $\text{tr}_{H \otimes H} d_{\alpha\omega} = \text{tr}_H \rho_\alpha \rho_\omega = 1$  in accord with (3.31), while  $\text{tr}_{\mathcal{H}} d_{\alpha\omega} = \sum_M w_M = \text{tr}_H \rho_\alpha \text{tr}_H \rho_\omega$ ; work it out directly or compare (3.42).)

### E. The ILS Theorem Revisited

One might be wondering what any of this has to do with the important ILS theorem (described in section IIB; *cf.* (2.9)) classifying decoherence functionals on  $\mathcal{P}(H)$ . As it turns out, what has been done amounts to an explicit, constructive (and elementary!) demonstration of the ILS theorem in finite dimensions. Moreover, we have also found a version of the ILS theorem applicable to decoherence functionals on class operators.

The recovery of the ILS theorem is not hard to see. Suppose we are considering a decoherence functional on  $\mathcal{R}_{\mathcal{P}}(H)$  (or  $\overline{\mathcal{R}}_{\mathcal{P}}(H)$ , or  $\mathcal{P}(H)$ .) (3.12) may then be rewritten

$$\langle h \| d \| h' \rangle = \text{tr}_{H \otimes H} [h \otimes h' \underline{d}]. \quad (3.32)$$

Comparing with (2.9), we discover that<sup>20</sup>

$$\underline{d} = X. \quad (3.33)$$

The property that  $X$  is normalized is just (3.31) above. That  $X^\dagger = M X M$  (where  $M$  is defined so that  $M(A \otimes B)M = B \otimes A$ ; see below (2.9)) is of course a consequence of the Hermiticity of  $d$  on  $\mathcal{H}$ , and is evident from the following calculation (*cf.* (3.17)):

$$\begin{aligned} \underline{d}^\dagger &= \left( \sum_{IJ} d_{IJ} E_I \otimes E_J^\dagger \right)^\dagger \\ &= \sum_{IJ} d_{JI}^* E_J^\dagger \otimes E_I \\ &= M \underline{d} M \end{aligned} \quad (3.34)$$

---

<sup>20</sup>To be strictly accurate, in theories with a time consistency with the Schrödinger picture usage of ILS requires the unitary time evolution operators to be “factored out” of the Heisenberg projections in  $h \otimes h'$  and absorbed into  $\underline{d}$  before making the identification with  $X$ . As  $h \otimes h'$  is simply a  $2-k$  fold tensor product of projections, the cyclicity of the trace makes this a straightforward procedure.

because  $d = d^*$  implies as usual that  $d_{IJ} = d_{JI}^*$ . That  $X$  is positive in the sense that  $\text{tr}[P \otimes P X] \geq 0$  is just the postulated positivity of  $d$  on  $\mathcal{R}$ . Unfortunately, it is not known how to capture this property more explicitly (for instance, as a transparent condition on the eigenvalues of  $d$  or  $\underline{d}$ .)

Thus, the ILS operator  $X$  emerges quite naturally in the present formulation of generalized quantum theory. From (3.12) it is clear that the correspondence between a decoherence functional  $d$  on  $\mathcal{H} = H \otimes H^*$  and  $\underline{d}$  on  $\underline{\mathcal{H}} = H \otimes H$  also has a more general utility. The most interesting observation for present purposes is that (3.12) *is effectively an expression of the ILS theorem applicable as well to decoherence functionals on class operators*: an arbitrary decoherence functional on class operators may be written as

$$d(h, h') = \text{tr}_{H \otimes H} [(h^\dagger \otimes h') \underline{d}], \quad (3.35)$$

thereby classifying decoherence functionals on class operators according to the operators  $\underline{d}$ . As in the case of the ILS operator  $X$ ,  $\underline{d}$  is Hermitian on  $\underline{\mathcal{H}}$ , and consequently obeys (3.17); is positive in the appropriate sense; and is normalized, (3.31).

Section VII contains further examples of why it is useful to remember this correspondence.

It is important to note that (3.32) is not an independent “proof” of the ILS theorem, because we had to appeal in section III A to a result of Wright [8] that guaranteed that the extension of the domain of a decoherence functional is well defined. This result was in turn a part of the proof of Wright’s extension of the ILS theorem. In return, however, the geometrical significance of the ILS theorem becomes evident.

## F. The Canonical Decoherence Functional as an Example

Finally, it is time to treat the canonical decoherence functional (2.17) on  $\overline{\mathcal{R}}_C$ . It is most convenient to work with the basis vectors (for  $\mathcal{H}$ )

$$\begin{aligned} \|M\rangle &\equiv R_M = R_{im} \\ &\equiv |i\rangle\langle m|. \end{aligned} \quad (3.36)$$

Here, the vectors (in  $H$ )  $|i\rangle$  and  $|m\rangle$  are eigenvectors of the initial and final conditions  $\rho_\alpha$  and  $\rho_\omega$ ,

$$\rho_\alpha = \sum_i a_i |i\rangle\langle i|, \quad (3.37)$$

$$\rho_\omega = \sum_m z_m |m\rangle\langle m|, \quad (3.38)$$

where of course the  $\{|i\rangle\}$  and  $\{|m\rangle\}$  are chosen to be orthonormal sets. In that case, the  $\{\|M\rangle\}$  constitute a complete orthonormal set in  $\mathcal{H}$ :

$$\begin{aligned} \langle M \| N \rangle &\equiv \langle im \| jn \rangle \\ &= \text{tr}_H |m\rangle\langle i | j\rangle\langle n| \\ &= \delta_{ij} \delta_{mn} \\ &\equiv \delta_{MN}; \end{aligned} \quad (3.39)$$



$$\begin{aligned}
\left( \sum_M \|M\rangle\langle M\| \right) \|A\rangle &= \sum_{im} (\text{tr}_H R_{im}^\dagger A) |i\rangle\langle m| \\
&= \sum_{im} |i\rangle\langle i| A |m\rangle\langle m| \\
&= A \\
&= \|A\rangle.
\end{aligned} \tag{3.40}$$

(Note also the computationally convenient fact that

$$\begin{aligned}
\sum_M \|M\rangle\langle M\| &= \sum_{im} |i\rangle\langle m| \otimes |m\rangle\langle i| \\
&= \sum_{im} |im\rangle\langle mi|;
\end{aligned} \tag{3.41}$$

*cf.* (3.18).)  $\{\|M\rangle\}$  is then an eigenbasis of  $d_{\alpha\omega}$ :

$$\begin{aligned}
\langle N \| d_{\alpha\omega} \| M \rangle &= \text{tr}_H \rho_\omega R_N^\dagger \rho_\alpha R_M \\
&= \langle m | \rho_\omega | n \rangle \langle j | \rho_\alpha | i \rangle \\
&= \sum_{im} a_i z_m \delta_{ij} \delta_{mn} \\
&= \sum_M w_M \delta_{MN}
\end{aligned} \tag{3.42}$$

(where it should be clear that  $M \equiv (im)$  and  $N \equiv (jn)$ ).

Thus a canonical decoherence functional may be written in the diagonal form

$$\begin{aligned}
d_{\alpha\omega} &= \sum_M w_M \|M\rangle\langle M\| \\
&= \sum_{im} a_i z_m \|im\rangle\langle im\|.
\end{aligned} \tag{3.43}$$

On  $\underline{\mathcal{H}} = H \otimes H$  this reads

$$d_{\alpha\omega} = \sum_{im} a_i z_m |im\rangle\langle mi| \tag{3.44}$$

using (3.36). (3.44) is the ILS operator  $\underline{d}$  of (3.35) for canonical decoherence functionals on class operators.

As  $\rho_\alpha$  and  $\rho_\omega$  are positive by definition,  $d_{\alpha\omega}$  is manifestly a positive operator on  $\mathcal{H}$ , and positive definite on the subspace corresponding to the non-zero eigenvalues of  $\rho_\alpha$  and  $\rho_\omega$ ,  $\mathcal{N}_{\alpha\omega}^\perp = \text{span}\{R_{im} \mid a_i \neq 0, z_m \neq 0\} \simeq \mathcal{H}_{\alpha\omega} \equiv \mathcal{H}/\mathcal{N}_{\alpha\omega}$ . In that case,  $\{\|\overline{M}\rangle\} = \{\overline{R}_M\} = \{(w_M)^{-\frac{1}{2}} R_M; w_M \neq 0\}$  is an orthonormal basis in the positive definite subspace  $\mathcal{H}_{\alpha\omega}$ , if  $d_{\alpha\omega}$  is taken to be the inner product on  $\mathcal{H}_{\alpha\omega}$ .

#### IV. SOME STRUCTURAL ISSUES IN GENERALIZED QUANTUM THEORY

An important issue in the program of generalized quantum theory is whether or not, and under what assumptions, it is possible to determine the “decoherence functional of the universe”, which question embraces the more familiar goals of both high energy physicists

(“What are the dynamics – lagrangian? – of the universe?”) and quantum cosmologists (“What are the boundary conditions of the universe?” See [49–52] for reviews of quantum cosmology.) That is, what is the minimum physical information required to (approximately) reconstruct  $d_{\text{universe}}$ ? While it is outside the scope of this paper to deal with this question directly, the study of the mathematical relations between decoherence functionals and decoherent histories is pertinent because these relations determine how much information about one set of objects may be inferred from information about the other.

In particular, within a fixed mathematical setting, it would be a good thing to have answers to at least the following questions. (Recall that  $\mathcal{C}_d$  is the class of exclusive, exhaustive sets of histories  $\mathcal{S}_d$  which are mutually consistent according to  $d$ , and  $\mathcal{D}_d$  the union of all such sets, *i.e.* the collection of histories  $h$  which appear in some  $\mathcal{S}_d \in \mathcal{C}_d$ .)

1. Given  $d$ , what is  $\mathcal{D}_d$ ?
2. Given  $d$ , what is  $\mathcal{C}_d$ ?
3. Given a history  $h$ , what are all the decoherence functionals  $d$  such that  $h \in \mathcal{D}_d$ ?
  - (a) What is the subset of such  $d$  according to which  $h$  is consistent, with a given probability  $p_h$ ?
4. Given an exclusive, exhaustive set of histories  $\mathcal{S}$ , what are all the decoherence functionals  $d$  such that  $\mathcal{S} \in \mathcal{C}_d$ ?
  - (a) What is the subset of all such  $d$  according to which  $\mathcal{S}$  is mutually consistent and given with a fixed set of probabilities  $\{p_h = d(h, h), h \in \mathcal{S} \in \mathcal{C}_d\}$ ?
5. Given a collection  $\{\mathcal{S}\}$  of exclusive, exhaustive sets of histories  $\mathcal{S}$ , what are all the decoherence functionals  $d$  for which  $\{\mathcal{S}\} \subseteq \mathcal{C}_d$ ?
  - (a) Given  $\{\mathcal{S}\}$  and a set of probabilities  $p_h$  on all the histories in this collection, what is the subset of decoherence functionals which give consistently  $\{p_h = d(h, h), h \in \mathcal{S} \in \{\mathcal{S}\} \subseteq \mathcal{C}_d\}$ ?
6. Does  $\mathcal{C}_d = \mathcal{C}_{d'}$  (or  $\mathcal{D}_d = \mathcal{D}_{d'}$ ) imply  $d = d'$ ? (Or is information about the probabilities required as well?)

At least in Hilbert space, something is known about all of these questions. Section V *formally* solves the first and second in finite dimensional Hilbert spaces by parameterizing the space of “histories” (operators) consistent according to a fixed decoherence functional, and showing moreover how – most explicitly in the case of positive decoherence functionals – to construct all of  $d$ ’s maximally fine-grained consistent sets. The practical issue that remains is determining just which of these consistent sets of operators contain only *physical* histories *i.e.* all of whose histories are in  $\mathcal{R}$  (*cf.* section IID). (An alternative would be to find a physical interpretation for Isham’s “exotic” projections, that is, the projections in  $\mathcal{P}(H)$  that are not in  $\overline{\mathcal{R}}_{\mathcal{P}}$ , or, in the case of  $\overline{\mathcal{R}}_{\mathcal{C}}$ , for the operators in  $\mathcal{O}_d$  that are not in  $\overline{\mathcal{R}}_{\mathcal{C}}$ .) Section VI shows how to characterize the decoherence functionals according to which a given set of histories is consistent, thereby solving questions 3 - 5 up to the issue of

unambiguously characterizing positivity on  $\mathcal{R}$  (again, only in finite dimensions.) The answer to question 6 is implicit in the observation that  $d$  is an Hermitian form, and is most certainly a (qualified) “yes”, though the precise form of this “yes” has yet to be determined. I do not take up the issue directly here. Variants on such questions are an interesting topic for future work.

Let me now show how to apply the tools of section II to these problems.

Though stated in general terms, much of the work of the following sections has the representation of histories by class operators primarily in mind. Part of the reason for this is that it is as class operators that histories are generally represented in actual applications of generalized quantum theory. The practical meaning of this emphasis is just that I shall not be making any particular use of the possibility that the history operators may be projections; the emphasis will be on  $d$  as a functional on operators, not the history operators themselves. (It is always possible to go back explore the consequences of the additional assumption that the history operators are projections. This would be of no particular consequence in section VI, where we get to select the histories *a priori*, but is a somewhat more significant issue in section V.)

## V. THE HISTORIES CONSISTENT WITH A DECOHERENCE FUNCTIONAL

This section is devoted to determining all of the histories that are consistent according to a given decoherence functional, *i.e.* determining  $\mathcal{D}_d$ . The next major section, VI, inverts this problem, determining all of the decoherence functionals according to which a given set of histories is consistent.

Though it is easy to characterize and parametrize the *operators* which satisfy the consistency condition, *i.e.* to find  $\mathcal{O}_d$ , the issues raised in section IID make it difficult to determine explicitly which of the operators in  $\mathcal{O}_d$  are actually members of  $\mathcal{D}_d$ .

Section VA discusses the characterization of  $\mathcal{O}_d$  and the difficulties in computing explicitly  $\mathcal{D}_d = \mathcal{O}_d \cap \mathcal{R}$ . Section VB displays one rather simple-minded way of parametrizing  $\mathcal{O}_d$ . In VC I make the observation that the eigenvectors of a decoherence functional may always be used to construct an exhaustive, consistent set of operators; section VD generalizes this observation to construct  $\mathcal{O}_d$  in another way: by  $d$ -unitary “rotations” of  $d$ ’s eigenvectors. (This strategy is easiest to implement for positive decoherence functionals like the canonical one of ordinary quantum mechanics.)

### A. The Condition for a History to be Consistent

If inhomogeneous coarse grainings are admitted,  $\{h, 1 - h\}$  is obviously the common coarsest graining that still contains  $h$  of all the exclusive, exhaustive sets which have  $h$  as a member. It is therefore natural to term  $h$  “consistent” according to  $d$  if this is a consistent set,  $\{h, 1 - h\} \in \mathcal{C}_d$  (though of course this is not meant to imply that *any* set in which  $h$

appears is consistent.) Thus, a history is consistent,  $h \in \mathcal{D}_d$ , only if <sup>21</sup>

$$d(h, 1 - h) = 0. \quad (5.1)$$

However, for an arbitrary operator in  $\mathcal{H} = \mathcal{L}(H)$ , (5.1) is *not* sufficient to imply that  $h \in \mathcal{D}_d$  in general, because there may be solutions to this equation in  $\mathcal{H}$  which do not lie in  $\mathcal{R}$ . Define the space of operator solutions to (5.1) as  $\mathcal{O}_d = \{h \in \mathcal{H} \mid d(h, 1 - h) = 0\}$ . The space of *physical* solutions is then  $\mathcal{D}_d = \mathcal{O}_d \cap \mathcal{R}$ , or,  $\mathcal{D}_d = \{h \in \mathcal{R} \mid d(h, 1 - h) = 0\}$ . Given the discussion in section IID, it is clear that  $\mathcal{O}_d$  is considerably more straightforward to work with than  $\mathcal{D}_d$ . In fact, the project of this section is to exhibit  $\mathcal{O}_d$  explicitly. Nevertheless, once the solutions to (5.1) have been found, additional criteria must then be applied to determine whether or not those solutions correspond to actual physical histories. (For instance, in the case of  $\mathcal{P}(H)$ , it must be checked which of the solutions are projections.) This is awkward, but so far unavoidable.

At present, the formal solution that  $\mathcal{D}_d = \mathcal{O}_d \cap \mathcal{R}$ , supplemented by the limited checks offered in section IID, is the best that can be made of the general situation.

Alternately, one could attempt to solve (5.1) just in  $\mathcal{R}_{\mathcal{P}}$  or  $\mathcal{R}_{\mathcal{C}}$  by employing an explicit parametrization of the projections which appear in (2.12) or (2.14), but significant problems remain when inhomogeneous coarse grainings are admitted. This does in any case generally yield equations quite impossible to actually solve. In the homogeneous case, Dowker and Kent [21] do briefly offer a – very formal – strategy for finding the solutions, but this is also generally impossible to carry out in practice.

Thus, while  $\mathcal{O}_d$  is very easy to find explicitly,  $\mathcal{D}_d$  is very hard. It is perhaps most profitable to view the situation this way: the problem of finding  $\mathcal{D}_d$  has been broken into two pieces, finding  $\mathcal{O}_d$ , and then determining whether  $h \in \mathcal{R}$ . If  $\mathcal{R} = \mathcal{P}(H)$  this is very easy, of course, but the other cases need more work.

(As an antidote to this negativity, it is perhaps worth emphasizing that it is always possible, at least in principle, to sort through  $\mathcal{R}$  or  $\overline{\mathcal{R}}$  and determine which of its members satisfy the consistency condition (5.1). It is just that we do not have the same nice characterizations of  $\mathcal{D}_d$  that are given for  $\mathcal{O}_d$  in sections VB and VD.)

While the inability of  $d$  to isolate elements of  $\mathcal{D}_d$  from  $\mathcal{O}_d$  may be annoying, it is hardly surprising. After all, in all cases a general Hilbert space decoherence functional extends naturally to an operator on the full space  $\mathcal{L}(H) = \mathcal{H}$ . Having done so,  $d$  is a fairly generic Hermitian form on  $\mathcal{H}$ . In fact, the only property of  $d$  which really “knows about” the underlying physical  $\mathcal{R}$  is (2.2), positivity of  $d$  on  $\mathcal{R}$ . (It is unfortunate that this property is difficult to express in terms less abstract than (2.2), say as a useful condition on the

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<sup>21</sup>It is worth emphasizing that (5.1) is a valid criterion for the consistency of  $h$  whether or not  $1 - h \in \mathcal{R}$ , and therefore whether or not inhomogeneous coarse grainings are admissible, a fact which should become obvious after a moment’s reflection on the fact that  $1 - h$  is the sum of all the other histories with which  $h$  is supposed to be consistent. This is good, because nothing requires  $1 - h$  to be in  $\mathcal{R}$  if  $h$  is. In particular, there is no guarantee whatever that  $1 - h$  is homogeneous if  $h$  is. In general it will not be, both in  $\mathcal{R}_{\mathcal{P}}$  and in  $\mathcal{R}_{\mathcal{C}}$ . This is yet another suggestion that it is somewhat unnatural from the histories point of view to exclude inhomogeneous coarse grainings.

eigenvalues of  $d$ . The exception, of course, is the case of  $\overline{\mathcal{R}_C}$ , where it is equivalent to positivity of  $d$  on  $\mathcal{H}$ .) Positivity of  $d$  on  $\mathcal{R}$  is clearly not enough to tell  $d$  what  $\mathcal{R}$  is.

Putting aside these limitations, the main goal of this section is to exhibit the solutions to (5.1) in  $\mathcal{H}$ , *i.e.* to find  $\mathcal{O}_d$ . This is the subject of the next subsection. Knowing  $\mathcal{O}_d$ , it is then possible to say something about how to build exhaustive sets of consistent histories (operators) from elements of  $\mathcal{O}_d$ , though as above the problem of determining the physical significance of these “histories” remains.

FIGURES

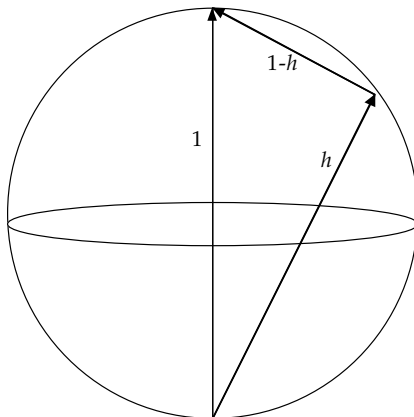


FIG. 1. Graphical representation of the consistency condition  $d(h, 1 - h) = 0$  for  $h$  to be a consistent history operator,  $h \in \mathcal{O}_d$ :  $h$  must be orthogonal to  $1 - h$  in the geometry defined by  $d$ .

To see how to build exhaustive consistent sets from  $\mathcal{O}_d$ , I make the following observation. Suppose  $h, h' \in \mathcal{D}_d$ . Then it is easy to check that  $\{h, h', 1 - h - h'\} \in \mathcal{C}_d$  iff  $d(h, h') = 0$ .<sup>22</sup> All we have to do to build exhaustive consistent sets, therefore, is to find in  $\mathcal{D}_d$  collections of  $d$ -orthogonal operators. I will show how to make use of this observation later in the section.

## B. Parametrization of $\mathcal{O}_d$

The space  $\mathcal{O}_d$  of solutions to the consistency condition  $d(h, 1 - h) = 0$  is easy to visualize. Rewrite (5.1) as  $d(h, h) = d(h, 1)$ .  $d(h, 1)$  is thus real for consistent  $h$ , and is moreover between 0 and 1 so long as  $h \in \mathcal{R}$ , or more generally, for any  $h \in \mathcal{H}$  so long as  $d$  is positive on  $\mathcal{H}$ ; see section IID. In any case, thinking of  $d$  as an inner product on the space of histories, (5.1) says that  $h$  is consistent,  $h \in \mathcal{O}_d$ , if its  $|\text{length}|^2$  in this inner product is equal to the length of its projection onto the direction  $1_H$ . If  $d$  is positive on  $\mathcal{H}$  (as for instance in the canonical case), or if  $h \in \mathcal{R}$ , the solutions to (5.1) may then be thought of as defining the surface of a round sphere (in the geometry defined by  $d$ ) which has its south pole at the origin and its north pole at the tip of the vector  $1_H$  (see figure 1).

This picture makes it easy to parameterize the solutions to the consistency condition explicitly. Find a basis  $\{B_I\}$  for  $\mathcal{H}_d = \mathcal{H}/\mathcal{N}_d$  ( $\dim \mathcal{H}_d = n_d$ ) which is orthonormal with respect to  $d$ , with  $B_1 = 1_H$ . (If  $d$  is positive, given a set of  $n_d$  linearly independent operators on  $\mathcal{H}_d$  that includes  $1_H$ , the usual Gram-Schmidt procedure using  $d$  suffices to construct such a set. If  $d$  is not positive on  $\mathcal{H}$ , a variant of the Gram-Schmidt argument shows a basis always exists for which  $d(B_I, B_I) = \pm 1$ , it being inconvenient for present purposes to allow “null” basis vectors,  $d(B_I, B_I) = 0$ .) The choice  $B_1 = 1_H$  is not critical, but it does simplify a little the following formulae.

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<sup>22</sup>To be fair, I am glossing over the question raised in the preceding footnote of whether or not  $1 - h - h' \in \mathcal{R}$ . Let us suppose either that inhomogeneous coarse grainings are permitted, or alternately, that we replace  $\mathcal{C}_d$  here by  $\mathcal{Q}_d$  defined in section VC below.

Writing  $h = \sum_{I=1}^{n_d} h_I B_I$  (*cf.* (3.15)), and setting  $\delta_I = d(B_I, B_I) = \pm 1$ , the consistency condition (5.1) becomes

$$h_1 = \sum_{I=1}^{n_d} \delta_I |h_I|^2. \quad (5.2)$$

Note that in this basis, (5.1) says that  $h_1$  is the probability of  $h$ ,  $d(h, h)$ . Rearranging slightly,

$$h_1^2 - h_1 + \sigma = 0 \quad (5.3)$$

where

$$\sigma = \sum_{I=2}^{n_d} \delta_I |h_I|^2; \quad (5.4)$$

$\sigma = \sum_{I=2}^{n_d} |h_I|^2$  for positive decoherence functionals, of course. (5.3) has real solutions for  $h_1$  so long as  $\sigma \leq \frac{1}{4}$ . (The two solutions for each  $\sigma < \frac{1}{4}$  obviously correspond to  $h$  and  $1-h$ .) Then  $0 \leq h_1 \leq 1$  (a pre-requisite for  $h$  to be in  $\mathcal{D}_d$ , and not just  $\mathcal{O}_d$ ) whenever  $0 \leq \sigma \leq \frac{1}{4}$ . If  $d$  is not positive, then it is possible for  $\sigma$  to be negative. So long as  $\sigma \leq \frac{1}{4}$ ,  $h$  is in  $\mathcal{O}_d$ , but when  $\sigma < 0$ ,  $h \notin \mathcal{D}_d$  because  $h_1 = d(h, h)$  is either bigger than 1 or less than zero.

For positive decoherence functionals (such as  $d_{\alpha\omega}$ ) the solution space to (5.1) is then just the interior of the sphere of radius  $\frac{1}{4}$  in  $\mathbb{C}^{n_d-1}$ . Otherwise the solution space is the interior of the “hyperboloid” in  $\mathbb{C}^{n_d-1}$  defined by  $\sigma \leq \frac{1}{4}$ . In this case, only the subset of parameters  $\{h_I \mid I = 2, \dots, n_d\}$  for which  $0 \leq \sigma \leq \frac{1}{4}$  have any hope of corresponding to *physical* solutions.

There is another approach to the problem of solving the consistency condition (5.1) which has some useful features. In order to discuss this solution, it is instructive to first fulfill a promise made at the beginning of this section.

### C. Eigenvectors of $d$ as a Consistent Set of Histories

I mentioned above that the eigenvectors of  $d$  may always be used to construct an exhaustive, consistent set of history operators. In this section I describe how this is done.

Of course, the discussion following (5.1) still applies. That is, there is no general guarantee that these operators in  $\mathcal{O}_d$  are actually in  $\mathcal{R}$ , *i.e. bona fide* physical histories. Fortunately, though, in the case of the canonical decoherence functionals it is manifest that they are. Nevertheless, in order to be accurate, I will as before denote the collection of exhaustive consistent *sets* of operators by  $\mathcal{Q}_d$ , so that  $\mathcal{C}_d$  comprises the subset of  $\mathcal{Q}_d$  which contains only *physical* consistent sets.<sup>23</sup> The distinction between  $\mathcal{Q}_d$  and  $\mathcal{C}_d$  is thus wholly analogous to that between  $\mathcal{O}_d$  and  $\mathcal{D}_d$ .

Suppose the decoherence functional has already been diagonalized as in (3.10),

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<sup>23</sup>“Exhaustive” here of course means that all the operators in a set sum to  $1_H$ .

$$d = \sum_I w_I \|I\rangle\langle I|, \quad (5.5)$$

where  $\{\|I\rangle\} \Leftrightarrow \{E_I\}$ . (For the purposes of this section, some of the  $w$ 's may be zero. The corresponding  $\|I\rangle$ 's are any normalized basis of  $d$ 's nullspace.) Define the vectors

$$\begin{aligned} \|D_I\rangle &= \|I\rangle\langle I| 1_H\rangle \\ &= \text{tr}_H E_I^\dagger \|I\rangle. \end{aligned} \quad (5.6)$$

Then the set  $\{\|D_I\rangle\}$  is an exhaustive consistent set,  $\{\|D_I\rangle\} \in \mathcal{Q}_d$ .

Indeed, it is clear from (5.6) that  $\{\|D_I\rangle\}$  is exhaustive,

$$\sum_I \|D_I\rangle = \|1_H\rangle. \quad (5.7)$$

(This is just the statement that  $1_H = \sum_I (\text{tr}_H E_I^\dagger) E_I$ .) It is also trivial to verify that the vectors  $\|D_I\rangle$  are mutually consistent. Individually, they all satisfy the consistency condition  $d(h, 1-h) = 0$ :

$$\begin{aligned} \langle D_I \| d \| D_I \rangle &\stackrel{?}{=} \langle 1_H \| d \| D_I \rangle \\ \langle D_I \| I \rangle w_I \langle I \| D_I \rangle &\stackrel{?}{=} \langle 1_H \| I \rangle w_I \langle I \| D_I \rangle \\ w_I |\langle I \| D_I \rangle|^2 &= w_I |\langle I \| D_I \rangle|^2 \end{aligned} \quad (5.8)$$

Thus each  $\|D_I\rangle \in \mathcal{O}_d$ . And, they are manifestly consistent with one another,

$$\begin{aligned} \langle D_I \| d \| D_J \rangle &= w_I |\langle I \| 1_H \rangle|^2 \delta_{IJ} \\ &= p_I \delta_{IJ}. \end{aligned} \quad (5.9)$$

For positive decoherence functionals, or if  $\|D_I\rangle \in \mathcal{R}$ ,  $d(1,1) = 1$  as usual forces (section IID) the “probabilities”  $p_I$  to satisfy  $0 \leq p_I \leq 1$ .

As promised, the eigenvectors of a decoherence functional always define an exhaustive, consistent set of history operators. (If none of the  $\langle I \| 1_H \rangle$  are zero, then this set even saturates the bound of section IIIB on the maximum number of histories with nonzero probability in a consistent set.) The possible physical interpretation of such histories is an issue to which I return in a moment.

In the case of the canonical decoherence functional (5.5) the eigenvectors are simply the  $\|im\rangle$  of (3.36), so that in this case (5.6) reads

$$\begin{aligned} \|D_{im}\rangle &= \text{tr}_H R_{im}^\dagger \|im\rangle \\ &= \langle i | m \rangle \|im\rangle. \end{aligned} \quad (5.10)$$

In this case, at least, the class operators (2.14) to which the  $\|D_{im}\rangle$  correspond are easy to find (*cf.* (3.36)):

$$\begin{aligned} \langle i | m \rangle R_{im} &= |i\rangle\langle i| m\rangle\langle m| \\ &= P_i P_m. \end{aligned} \quad (5.11)$$



Thus  $\{\|D_{im}\rangle\} \in \mathcal{C}_d$  (not just  $\mathcal{Q}_d$ ).

For more general decoherence functionals, however, it is not at all clear that the  $\{\|D_I\rangle\}$  do, in fact, correspond to any *physical* histories, that is, that  $\{\|D_I\rangle\} \in \mathcal{C}_d$ , as befits the discussion following (5.1). For instance, for a randomly selected decoherence functional on  $\mathcal{R}_P$ , the eigenvectors of  $d$  will rarely turn out to be projections. (As a fairly arbitrary Hermitian operator on  $\mathcal{H} = \mathcal{L}(H)$ ,  $d$  has no way of knowing that it is only supposed to see projection operators.) In the general case, we must simply check (as in section IID) whether or not  $\{\|D_I\rangle\} \in \mathcal{C}_d$  in order to determine whether or not the eigenvectors of  $d$  have more than mere mathematical significance.

#### D. $d$ -Unitarity and Consistent Sets of Operators

Now that we have found one consistent set of history operators, the aim of this part is to find the rest of them. Just as all the orthonormal bases of a vector space can be obtained from some fixed basis by unitary transformations, we can obtain the other maximally fine grained elements of  $\mathcal{Q}_d$  from the eigenvectors of  $d$  by employing  $d$ -unitary transformations, followed by a rescaling that guarantees each new mutually consistent set still sums to  $|1_H\rangle$  – *i.e.* is still exhaustive. The complete recipe for this procedure is exhibited explicitly for positive decoherence functionals

In the case of positive decoherence functionals, the set of  $d$ -unitary transformations that map one exhaustive consistent set into another set with the same probabilities will turn out to be just  $SU(n-1)$ . More generally, *every*  $d$ -unitary transformation maps one exhaustive consistent set into another, so long as the “rotation” is followed by an appropriate rescaling of the probabilities. This rescaling is fixed by the rotation, so the freedom to generate new consistent sets from old turns out to be  $SU(n_+, n_-)$ , where  $n_+, n_-$  are the number of  $d$ ’s positive and negative eigenvalues, respectively.

The techniques employed here may also be of some use in studying, for instance, the symmetries of decoherence functionals, a subject I plan to take up elsewhere.<sup>24</sup>

To begin, I define the notion of  $d$ -unitarity, before moving on to describe how to use  $d$ -unitary transformations to obtain one consistent set from another. In order to simplify the discussion a bit I will assume that  $\mathcal{N}_d = \emptyset$ . In addition, I drop the  $\odot$  notation of (3.22) because it is cumbersome here, and  $\underline{H} = H \otimes H$  plays no role in what follows.

##### $d$ -Unitarity

An invertible linear transformation  $V$  on  $\mathcal{H}$  will be said to be  *$d$ -unitary* if

$$d(Vh, Vh') = d(h, h') \quad \forall h, h' \in \mathcal{H}, \quad (5.12)$$

in which case the condition for  $d$ -unitarity is

$$d = V^* d V. \quad (5.13)$$

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<sup>24</sup>Some very interesting work on this topic has recently appeared in [53,54].

As  $|\det V| = 1$  and any overall phase is clearly irrelevant, we can always take  $\det V = 1$ . Next, define  $W$  and  $\delta$  via the “polar decomposition” of  $d$  in  $\mathcal{H}$  (cf. (7.27)),

$$d = \delta W, \quad (5.14)$$

where

$$\begin{aligned} W &= |d|_{\mathcal{H}} \\ &= \sum_I |w_I| \|I\rangle\langle I\| \end{aligned} \quad (5.15)$$

and

$$\delta = \sum_I \delta_I \|I\rangle\langle I\|. \quad (5.16)$$

The notation here is the same as in (5.5) above, of course, and  $\delta_I = \text{sign}(w_I) = \pm 1$ . If  $d$  is positive on  $\mathcal{H}$ ,  $\delta$  is of course just  $1_{\mathcal{H}}$ , and  $d = W$ . (Remember that  $\delta$  is severely restricted by the facts that  $d$  is not only positive on  $\mathcal{R}$ , but also – see section III C – on the span of any would-be physical consistent set.)

Notice that  $\delta = d W^{-1}$ . Inserting factors of  $\sqrt{W}$  appropriately into (5.13), it is easy to see that it is always possible to write  $V$  as<sup>25</sup>

$$V = \sqrt{W^{-1}} U \sqrt{W}, \quad (5.17)$$

where  $U$  satisfies

$$U^* \delta U = \delta. \quad (5.18)$$

In other words,  $U \in SU(n_+, n_-)$ , where  $n_+$  and  $n_-$  count the number of  $d$ ’s positive and negative eigenvalues, respectively, so  $n_+ + n_- = n = \dim \mathcal{H}$ . (For information on  $SU(p, q)$  see [55, 56].) When  $d$  is positive on  $\mathcal{H}$ ,  $\delta = 1$  and  $U$  is therefore just a unitary operator.

### *New Consistent Sets from Old*<sup>26</sup>

Following the lead of section V C, let me now describe how to employ  $d$ -unitary maps to generate exhaustive, consistent sets of operators from  $d$ ’s eigenvectors, the  $\{\|I\rangle\}$  of (5.5).

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<sup>25</sup>If  $\mathcal{N}_d \neq \emptyset$ , then  $V$  is the direct sum of an operator of the form (5.17) on  $\mathcal{H}_d \simeq \mathcal{N}_d^\perp$  and an invertible operator on  $\mathcal{N}_d$ . This is because a  $d$ -unitary  $V$  leaves  $\mathcal{N}_d$  and  $\mathcal{N}_d^\perp$  invariant. To see this, note that (5.13) still holds on all of  $\mathcal{H}$ . If  $V$  mapped  $\mathcal{N}_d^\perp \rightarrow \mathcal{N}_d$  then there would be an  $\|h\rangle \in \mathcal{N}_d^\perp$  for which  $d\|h\rangle \neq 0$  but  $dV\|h\rangle = 0$ , violating (5.13). Conversely, if  $V$  mapped  $\mathcal{N}_d \rightarrow \mathcal{N}_d^\perp$  then there would be some  $\|h\rangle \in \mathcal{N}_d$  for which  $d\|h\rangle = 0$  but  $dV\|h\rangle \neq 0$ . Insisting that  $V$  be invertible means that  $V^*dV\|h\rangle \neq 0$  as well, again contradicting (5.13).  $V$  thus splits up into (5.17) on  $\mathcal{H}_d \simeq \mathcal{N}_d^\perp$ , and an invertible piece in  $\mathcal{N}_d$  that more or less just “comes along for the ride” in the subsequent discussion.

<sup>26</sup>I am grateful to Tomáš Kopf for very useful conversations concerning the material of this section.

For positive decoherence functionals it will be clear that we can get *all* of the maximally fine grained, exhaustive, sets of operators consistent according to  $d$  in this way. (More work is required for decoherence functionals which are not positive.)

It is of course true that if  $\{h\}$  is a consistent set of operators, then  $d(h, h') = 0$  when  $h \neq h'$ , and naturally the set  $\{Vh\}$  will have the same property. However, it will not generally be true that  $\{Vh\}$  is a *consistent* set because a general  $d$ -unitary “rotation” will move each  $h$  off of the “consistency sphere” pictured in figure 1, so that  $Vh \notin \mathcal{O}_d$  and  $\{Vh\} \notin \mathcal{Q}_d$ .

There is a simple remedy for this, however:  $Vh$  can be rescaled to put its tip back on to the “consistency sphere”. Consider a consistent  $h \in \mathcal{O}_d$  and an arbitrary  $d$ -unitary  $V$ . We would like to rescale  $Vh$  by a number  $t$  so that  $tVh$  is consistent as well:

$$d(tVh, 1) = d(tVh, tVh) \quad (5.19)$$

so that

$$t_h = \frac{d(Vh, 1)}{d(h, h)}. \quad (5.20)$$

This is just the coefficient of  $h$  in the expansion of  $V^{-1}1$  in the  $d$ -orthogonal set  $\{h\}$ ; the significance of this observation will become evident in (5.22)-(5.24) below. (Also compare (5.6).) The consistency of  $h$  implies that  $t$  is real; it obviously depends on both  $h$  and  $V$ .

(If  $h$  has zero probability,  $d(h, h) = 0$ , then  $t^* d(Vh, 1) = 0$  in order that  $tVh$  be consistent, so that  $t = 0$  if  $d(Vh, 1) \neq 0$ . If  $d(Vh, 1) = 0$ , it may be verified that the correct  $t$  to guarantee exhaustiveness – *i.e.* (5.24) – is again the coefficient of  $h$  in the expansion of  $V^{-1}1$  in the set  $\{h\}$ . This is always possible so long as  $\text{span}\{h\} = \mathcal{H}$ .)

The map  $\{h\} \rightarrow \{t_h Vh\}$  generates new consistent sets from old. Note, however, that it is sufficient that the initial set  $\{h\}$  be mutually  $d$ -orthogonal in order that  $\{t_h Vh\}$  be a genuinely consistent set of operators, *i.e.* be in  $\mathcal{Q}_d$ ; we did not have to require that the  $\{h\}$  sum to 1. (The consistency of  $h$  only shows that  $t_h$  is real, an inessential property for present purposes.)

In order to construct exhaustive, consistent sets of operators by this “rotate-rescale” procedure, it is therefore convenient to begin with – switching now to history space Dirac notation –  $d$ ’s eigenvectors  $\{\|I\rangle\}$ . They comprise a normalized, linearly independent set of  $d$ -orthogonal operators, and none of them have  $\langle I \| d \| I \rangle = 0$  when  $\mathcal{N}_d = \emptyset$ , as we are here assuming.

With the choice (5.20) of  $t_I$  for each  $\|I\rangle$ ,  $\|\tilde{I}\rangle \in \mathcal{O}_d$  and  $\{\|\tilde{I}\rangle\} \in \mathcal{Q}_d$ , where

$$\|\tilde{I}\rangle = t_I V \|I\rangle. \quad (5.21)$$

In other words, the  $\{\|\tilde{I}\rangle\}$  comprise an exhaustive, consistent set of operators.

Indeed, from the  $d$ -unitarity of  $V$  it is obvious that the  $\|\tilde{I}\rangle$  are mutually  $d$ -orthogonal. In order to verify that  $\{\|\tilde{I}\rangle\}$  is exhaustive, we need to show that  $\sum_{\tilde{I}} \|\tilde{I}\rangle = \|1_H\rangle$ :

$$\sum_{\tilde{I}} \|\tilde{I}\rangle = \sum_I \frac{d(VI, 1)}{d(I, I)} V \|I\rangle$$

$$\begin{aligned}
&= V \sum_I \frac{\langle I \| V^* d \| 1_H \rangle}{\langle I \| d \| I \rangle} \| I \rangle \\
&= V \sum_I \frac{\| I \rangle \langle I \|}{\langle I \| d \| I \rangle} V^* d \| 1_H \rangle \\
&= V \sum_I \frac{\| I \rangle \langle I \| d}{\langle I \| d \| I \rangle} V^{-1} \| 1_H \rangle,
\end{aligned} \tag{5.22}$$

where the last line follows from the  $d$ -unitarity of  $V$ ,  $d = V^* d V$ .

Recall now that the  $\{\| I \rangle\}$  are a linearly independent set of vectors – in other words, a basis for  $\mathcal{H}$ . Because they are also  $d$ -orthogonal,

$$\sum_I \frac{\| I \rangle \langle I \| d}{\langle I \| d \| I \rangle} \| I' \rangle = \| I' \rangle. \tag{5.23}$$

As  $\{\| I \rangle\}$  is a basis, this means that

$$\begin{aligned}
\sum_I \| \tilde{I} \rangle &= V V^{-1} \| 1_H \rangle \\
&= \| 1_H \rangle
\end{aligned} \tag{5.24}$$

and  $\{\| \tilde{I} \rangle\}$  is indeed exhaustive.

Thus, with the rescalings appropriate to keep rotated vectors on the consistency sphere of figure 1,  $d$ -unitary “rotations” can generate exhaustive consistent sets from the eigenvectors of  $d$ . In the case of positive decoherence functionals, the operator  $U$  appearing in (5.17) is unitary, and it is geometrically clear that we can generate in this way *all* of the maximally fine grained, exhaustive sets of operators consistent according to  $d$ ; coarse grainings of these sets generate  $\mathcal{Q}_d$ . (There is obviously a great deal of redundancy in this description of  $\mathcal{Q}_d$ .) As ever, the question of which of these sets are physical remains.

We have to be somewhat more careful in the case of decoherence functionals which are not positive. The reason for this is the following: our map  $\{I\} \rightarrow \{t_I V I\}$  can only produce trivial zero probability consistent histories ( $t_I = 0$ ); a  $d$ -unitary transformation can *never* map a zero probability history to a non-zero probability one. However, even when  $\mathcal{N}_d = \emptyset$  (as I have assumed for convenience in this section) there are, in general, non-trivial zero probability history operators that are consistent according to a non-positive  $d$ .

I suspect it is possible to generate all of the maximally fine grained exhaustive consistent sets of a non-positive decoherence functional by the rotate-rescale transformations  $\{h\} \rightarrow \{t_h V h\}$ , so long as we begin with an appropriate collection of  $d$ -orthogonal sets  $\{h\}$ , each with a different number of zero probability histories. However, because of the non-trivial “light-cone” structure endowed on  $\mathcal{H}$  by a non-positive  $d$ , this appears to be a complicated issue to address in general.

It is of course true that applying the rescaling (5.20) to every complete, linearly independent  $d$ -orthogonal set  $\{h\}$  will yield all of the maximally fine-grained consistent sets of  $d$  (up to degeneracy in linearly dependent, zero-probability histories). It is just that we do not have the same nice classification of all of the maximal  $d$ -orthogonal sets by the operators  $V$  that we do in the positive case.

(The proof that a unitary operator always exists which maps one orthonormal basis into any other effectively identifies ordered orthonormal bases with unitary operators, given a

fiducial basis. For positive decoherence functionals, the more or less obvious generalization of this proof puts  $d$ -orthogonal sets in correspondence with  $d$ -unitary operators, given a fiducial  $d$ -orthogonal set –  $d$ 's eigenvectors – whence the fact that the “rotate-rescale” transformations are sufficient to generate all of the maximally fine-grained consistent sets when  $d$  is positive. On the other hand, the bases of a complex vector space that are orthogonal in a metric of indefinite signature (the  $\delta$  of (5.16)) are *not* in correspondence with  $U$ 's in  $SU(n_+, n_-)$  ( $U$ 's satisfying (5.18)) in the same simple way. This is just the observation I made above that a non-zero probability history can never be mapped to a zero probability history by a  $d$ -unitary  $V$ . All that this means is that another classification of the  $d$ -orthogonal sets must be employed. I delve into the issue no further here.)

### *A Nice Picture*<sup>27</sup>

For positive decoherence functionals, there is a very nice picture of these rotate-rescale transformations that is based on the “consistency sphere” of figure 1. Recall that this figure is drawn in in the geometry defined by  $d$ . In that geometry, consistent history operators are just those which lie on the sphere surrounding the preferred vector  $1_H$ . We would like to generalize the picture to accommodate consistent sets with more than two histories in them. Therefore, transplant the vector  $1 - h$  so that its base is at the origin (south pole); its tip of course still lies on the sphere. Now, as noted at the end of section V A, if  $h, h' \in \mathcal{O}_d$ ,  $\{h, h', 1 - h - h'\} \in \mathcal{Q}_d$  iff  $d(h, h') = 0$ . Figure 1 may therefore be generalized for  $\{h, h', 1 - h - h'\}$  to figure 2. (Three mutually ( $d$ -)orthogonal histories is the most we can draw on paper, but the generalization to more histories (higher dimensions) is clear.)

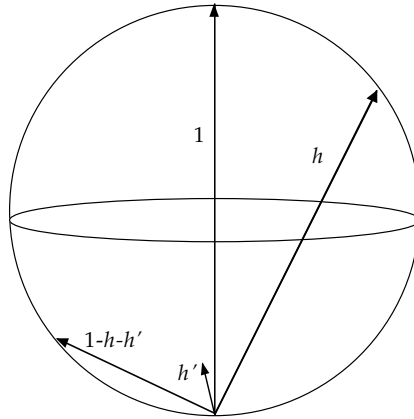


FIG. 2. Members of an exhaustive consistent set  $\{h, h', 1 - h - h'\}$  must be mutually orthogonal in the geometry defined by  $d$ , and they must all lie on the “consistency sphere” defined by (5.1).

The consistency preserving  $d$ -unitary rotation-plus-rescaling transformations described in the previous subsection merely correspond to rigid rotations of the  $d$ -orthogonal set  $\{h, h', 1 - h - h'\}$ , while simultaneously stretching or squeezing each vector so that its tip remains on

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<sup>27</sup>I would like to thank Tomáš Kopf for the conversations which led to the figures of this section.

the surface of the consistency sphere, as pictured in figure 3. (The probability-preserving maps described in the next subsection are just those which spin the set rigidly “around” the vector 1.)

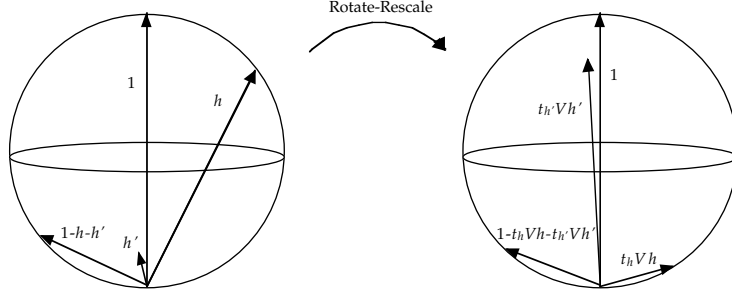


FIG. 3. The rotation-plus-rescaling transformation of one exhaustive consistent set into another.

### *Probability Preserving Transformations*

There is a special subclass of these transformations that is of some interest, namely, those which preserve the probabilities of the original set. These are just those  $V$ 's for which all the  $t_h = 1$ , so – dropping the sub- $H$  on the unit 1 in  $H$  – that  $1 = \sum Vh = V(\sum h) = V1$ . In other words, we also need

$$V \|1\rangle = \|1\rangle. \quad (5.25)$$

In terms of  $U$  this reads

$$U \|\tilde{1}\rangle = \|\tilde{1}\rangle, \quad (5.26)$$

where

$$\|\tilde{1}\rangle \equiv \sqrt{W} \|1\rangle. \quad (5.27)$$

(Note that when  $d$  is positive,  $W = d$ , so that  $\langle \tilde{1} \| \tilde{1} \rangle = \langle 1 \| d \| 1 \rangle = 1$ . Otherwise,  $\langle \tilde{1} \| \delta \| \tilde{1} \rangle = 1$  in the same way.)

The collection of  $d$ -unitary maps which preserve both the probabilities and the exhaustiveness of a set of histories is thus specified by the subgroup of  $SU(n_+, n_-)$  which has  $\|\tilde{1}\rangle$  as a fixed point.

At least in the case of the positive decoherence functionals, the condition (5.26) can be solved explicitly. To see how, note that  $U$ 's which satisfy (5.26) may always be written as

$$U = \|\tilde{1}\rangle \langle \tilde{1}| + U_\perp, \quad (5.28)$$

where (5.26) and unitarity imply that

$$\begin{aligned} U_\perp \|\tilde{1}\rangle &= U_\perp^* \|\tilde{1}\rangle \\ &= 0, \end{aligned} \quad (5.29)$$

and so

$$\begin{aligned}
U_{\perp}^* U_{\perp} &= 1 - \|\tilde{1}\rangle\langle\tilde{1}\| \\
&= 1_{\perp}.
\end{aligned}
\tag{5.30}$$

Thus, the maps which preserve both the probabilities and the completeness of a set of consistent histories are completely covered by  $SU(n-1)$  when  $d$  is positive.

In the general case where  $\delta \neq 1$ , a formulation very similar to the positive case can be worked out, but the solution does not come out nearly so nicely. The basic reason for this is that  $\|\tilde{1}\rangle$  is never an eigenvector of  $\delta$ , and thus the condition that  $\|\tilde{1}\rangle$  be an eigenvector of  $U$  does not break the symmetry down to an  $SU(p, q)$  subgroup of  $SU(n_+, n_-)$  in a clean way. What you get seems no easier to use than (5.26), and I therefore won't bother to discuss it.

## VI. THE DECOHERENCE FUNCTIONALS CONSISTENT WITH A SET OF HISTORIES

The aim of this section is to address some of the mathematical aspects of the problem of determining how observations constrain the decoherence functional of a closed quantum system. To that end, I show how to construct all of the decoherence functionals according to which a fixed set of histories is consistent in section VIB. I also briefly address the question of how to use these results to determine the family of decoherence functionals according to which a *collection* of sets of histories are all consistent.

### A. The Problem

While it is always possible to make physical predictions *given* a decoherence functional, it is clearly of physical interest to know how observations constrain what the possible decoherence functionals of a closed system might be. A first simple step is to determine what are all the decoherence functionals according to which a given history or set of histories is consistent. The aim of this section is to approach this problem when the observables of the system live in a finite dimensional Hilbert space.

Here, at least, the difficulty in determining when an operator in  $\mathcal{H}$  represents a physical history is not an issue because we are picking histories and finding decoherence functionals, instead of the other way around.

An explicit, constructive solution to this problem is given in what follows. Namely, given an exhaustive set of histories and their probabilities, it is shown in (6.7) how to construct the decoherence functionals according to which this set is consistent, and which give the required probabilities. (The practical issue that remains when  $d$  is not *a priori* positive is how to characterize operationally the positivity of  $d$  on the elements of  $\mathcal{R}$  which do not lie in the span of the given set of histories.)

## B. Decoherence Functionals Consistent with Sets of Histories

By now, the method for constructing decoherence functionals according to which a given set of histories is consistent is almost embarrassingly obvious.<sup>28</sup> It is based on the simple observation that a linear operator is determined by its matrix elements. (The construction of this section is very much in the spirit of that of Schreckenberg [7] for the case  $\mathcal{R} = \mathcal{P}(H)$ .)

Suppose that we have a set of disjoint, linearly independent histories  $\mathcal{S} = \{\|K\rangle\}$ , and a set of “probabilities”  $\{p_K\}$  for those histories. (Remember from section III B that all the non-zero probability histories in a consistent set must be linearly independent. Therefore, coarse grain all the  $p_K = 0$  histories into one.) It is not necessary at this point to assume that  $\mathcal{S}$  is exhaustive, *i.e.* that  $\sum_K \|K\rangle = \|1_H\rangle$  and  $\sum_K p_K = 1$ . Nor is it necessary to suppose that  $\mathcal{H}_\mathcal{S} = \mathcal{H}$ , where I am calling  $\text{span } \mathcal{S} = \mathcal{H}_\mathcal{S}$ . We will see later what we get for assuming that  $\mathcal{S}$  is exhaustive and/or maximally fine grained.

The problem is to find the decoherence functionals according to which the set  $\mathcal{S}$  is consistent. First the decoherence functionals according to which the histories in  $\mathcal{S}$  are mutually  $d$ -orthogonal will be found; when  $\mathcal{S}$  is exhaustive we will see that  $\mathcal{S} \in \mathcal{C}_d$  for free (*i.e.* that all the  $\|K\rangle \in \mathcal{D}_d$ ). Another way to state the problem is, what are all the  $d$  for which

$$\langle K \| d \| K' \rangle = p_K \delta_{KK'}. \quad (6.1)$$

But the  $\{\|K\rangle\}$  are linearly independent, so (6.1) *defines* the restriction of  $d$  to  $\mathcal{H}_\mathcal{S}$ . To  $d|_{\mathcal{H}_\mathcal{S}}$  may be added any Hermitian operator on  $\mathcal{H}$  which maps  $\mathcal{H}$  to  $\mathcal{H}_\mathcal{S}^\perp$  – in other words, any operator with zero matrix elements in  $\mathcal{H}_\mathcal{S}$  – that keeps  $d$  positive on  $\mathcal{R}$ , generating the full family of decoherence functionals which satisfy (6.1).

To construct  $d|_{\mathcal{H}_\mathcal{S}}$  explicitly, first find an orthonormal basis  $\{\|M\rangle\}$  in  $\mathcal{H}_\mathcal{S}$ . (If  $\mathcal{R} = \mathcal{P}(H)$ , disjoint projections are already trace-orthogonal and it is sensible to choose  $\|M\rangle = \|K\rangle / \sqrt{\langle K \| K \rangle}$ . Disjoint class operators, on the other hand, are rarely trace-orthogonal.) Then we may expand

$$\begin{aligned} \|K\rangle &= \sum_M \|M\rangle \langle M \| K \rangle \\ &= \sum_M W_{MK} \|M\rangle \end{aligned} \quad (6.2)$$

and

$$\|M\rangle = \sum_K V_{KM} \|K\rangle. \quad (6.3)$$

(If  $\|K\rangle \propto \|M\rangle$ ,  $W_{MK}$  and  $V_{KM}$  are obviously just proportional to  $\delta_{KM}$ .) It is easy to check that

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<sup>28</sup>A better trick would be to construct the decoherence functionals according to which *several* sets of histories  $\{\mathcal{S}_1, \dots, \mathcal{S}_m\}$  are individually consistent, *i.e.*  $\{d \| \{\mathcal{S}_1, \dots, \mathcal{S}_m\} \subset \mathcal{C}_d\}$ , but I address that problem only briefly here.



$$\sum_M V_{K'M} W_{MK} = \delta_{KK'} \quad (6.4)$$

and

$$\sum_M W_{M'K} V_{KM} = \delta_{MM'}, \quad (6.5)$$

so that the  $V_{KM}$  may be calculated as the matrix inverse to the known quantities  $W_{MK} = \langle M \| K \rangle$ . (If  $p_K \neq 0$  then it is clear from (6.1) and (6.3) that  $V_{KM} = \langle K \| d \| M \rangle / p_K$ .)

It is convenient to expand the unit on  $\mathcal{H}_S$  as

$$\begin{aligned} 1_{\mathcal{H}_S} &= \sum_M \|M\rangle \langle M| \\ &= \sum_{MK} V_{KM}^* \|M\rangle \langle K|. \end{aligned} \quad (6.6)$$

With (6.6) it is easy to reconstruct  $d|_{\mathcal{H}_S}$  from (6.1) by multiplying on the left by  $V_{KM}^* \|M\rangle$ , on the right by  $V_{K'M'} \langle M'|$ , and summing over  $KK'MM'$ . You get

$$\begin{aligned} d|_{\mathcal{H}_S} &= \sum_{MM'} d_{MM'} \|M\rangle \langle M'| \\ &= \sum_{MM'} \left( \sum_K p_K V_{KM}^* V_{KM'} \right) \|M\rangle \langle M'|, \end{aligned} \quad (6.7)$$

where

$$d_{MM'} = \sum_{KK'} V_{KM}^* d_{KK'} V_{K'M'}, \quad (6.8)$$

$d_{KK'}$  being defined by (6.1). The members of  $\mathcal{S}$  are mutually  $d$ -orthogonal according to this decoherence functional. When  $\mathcal{S}$  is exhaustive,  $\sum_K \|K\rangle = \|1_H\rangle$  and  $\sum_K p_K = 1$ , it is easy to check that each  $\|K\rangle \in \mathcal{D}_d : d(K, 1) = d(K, \sum_{K'} K') = \sum_{K'} d(K, K') = d(K, K)$ , so that  $\mathcal{S} \in \mathcal{C}_d$ . A similarly short calculation shows that this  $d$  is also properly normalized,  $d(1, 1) = 1$ .

As already noted, we can just add to  $d|_{\mathcal{H}_S}$  any Hermitian operator on  $\mathcal{H}$  which has zero matrix elements in  $\mathcal{H}_S$  to generate all the decoherence functionals for which  $\mathcal{S} \in \mathcal{C}_d$ . If  $\mathcal{S}$  is maximally fine grained (*i.e.* possesses  $\dim \mathcal{H}$  members; *cf.* section III B), then  $\mathcal{H}_S = \mathcal{H}$  and  $d$  is specified uniquely by  $(\mathcal{S}, \{p_K\})$ , hardly a surprising result.

I have so far said nothing about positivity. That  $d|_{\mathcal{H}_S}$  is positive on  $\mathcal{H}_S$  should be clear from (6.1) and the fact that  $\{\|K\rangle\}$  spans  $\mathcal{H}_S$ . If  $\mathcal{H}_S \neq \mathcal{H}$ , the freedom to add operators to  $d|_{\mathcal{H}_S}$  is clearly restricted by the requirement that  $d$  is positive on  $\mathcal{R}$ ; it is unfortunate that this property is difficult to characterize constructively.

Notice from this that *decoherence functionals are positive on the span of each of their (physical) consistent sets*. Thus any decoherence functional which possesses a consistent set that spans  $\mathcal{H}$  (a maximally fine grained consistent set with no zero probability members) is positive on all of  $\mathcal{H}$ . (Compare section III C.)

Finally, consider briefly the problem of determining the family of decoherence functionals according to which a collection  $\{\mathcal{S}_1, \dots, \mathcal{S}_m\} \subset \mathcal{C}_d$ .<sup>29</sup> Suppose we are given the matrix elements of  $d$  on each  $\mathcal{S}_i$ . This specification must of course be given self-consistently on each intersection  $\mathcal{H}_{\mathcal{S}_i} \cap \mathcal{H}_{\mathcal{S}_j}$ . The remaining freedom to specify  $d$  lies in the matrix elements *between* the  $\mathcal{H}_{\mathcal{S}_i}$ , and in the complement to span  $\{\mathcal{S}_1, \dots, \mathcal{S}_m\}$ . To say anything much more explicit requires some detailed information about the  $\mathcal{S}_i$ .

Suppose therefore that some of the consistent sets of histories of a closed system  $S$  have somehow been determined. On the assumption that  $S$  is described by a decoherence functional (generalized quantum state)  $d$ , the decoherence functional of  $S$  may be reconstructed on the span of these consistent sets just as above; the matrix elements between the consistent sets are left as free parameters, restricted only by the requirement that  $d$  is positive on  $\mathcal{R}$ . (If the probabilities have not also been found they may be regarded as additional free parameters in  $d$ . If so, they are restricted by the requirement that  $d$  is consistently defined on each intersection  $\mathcal{H}_{\mathcal{S}_i} \cap \mathcal{H}_{\mathcal{S}_j}$ .)

(There is a curious physical problem underlying all of this: how does one determine *any* consistent set of histories of a closed system, since only one history will actually be realized? The answer is that one doesn't. Rather, one sorts through the observed properties of, say, the universe, and takes the set of histories corresponding to each property  $P$ ,  $(h_P, 1 - h_P)$ , to be consistent. In other words, for each observed property  $P$ ,  $h_P$  must be a coarse-graining of the "one true history" of our universe. Taken together, the observed properties of the universe thereby constrain the possible decoherence functionals which may describe its state. I hope to return to this issue in a future publication.)

## VII. WHEN IS A DECOHERENCE FUNCTIONAL CANONICAL?

### A. Two Criteria

In section III F I showed how the canonical decoherence functional  $d_{\alpha\omega}$  of (2.17) defines an inner product on  $\mathcal{H}_{\alpha\omega} \equiv \mathcal{H}/\mathcal{N}_{\alpha\omega}$ . In this section the converse problem is posed: when does a positive Hermitian form on  $\mathcal{H} \simeq H \otimes H^*$  define a canonical decoherence functional? This investigation only addresses the issue for decoherence functionals defined on the class operator representation  $\mathcal{R}_C(H)$  of histories in the Hilbert space  $H$ <sup>30</sup>; the solution is expressed

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<sup>29</sup>Sadly, though all such  $d$  are positive on each of the  $\mathcal{H}_{\mathcal{S}_i}$ , this does not necessarily imply they are all positive on the join of these spaces. A simple example illustrates why. Suppose  $d = A \|A\rangle\langle A| - C \|C\rangle\langle C|$ , where  $\|A\rangle$  and  $\|C\rangle$  are unit, and  $\langle A \| C \rangle = 0$ . Then  $\langle B \| d \| B \rangle > 0$  so long as  $A/C > \frac{|\langle C \| B \rangle|^2}{|\langle A \| B \rangle|^2}$ . In that case,  $d$  is positive on span  $\{\|A\rangle\}$  and span  $\{\|B\rangle\}$ , but not on span  $\{\|A\rangle, \|B\rangle\}$ .

<sup>30</sup>As I mentioned in section II B, it is straightforward (if rather messy) to find explicitly the form of the ILS operator  $X_{\alpha\omega}$  on  $\otimes^{2k} H$  corresponding to a canonical decoherence functional (2.17) on  $\overline{\mathcal{R}}_C(H)$ . Given an ILS operator  $X$  on  $\otimes^{2k} H$  it is therefore possible to determine whether it is canonical by checking whether its matrix elements have the proper form. This discussion would

in (7.8) or (7.9) below.

To pose the question more explicitly, given some positive decoherence functional  $d$  over  $\mathcal{R}_C$  or  $\overline{\mathcal{R}}_C$ , when may it be written in the form (3.43)

$$\begin{aligned} d &= \sum_{im} a_i z_m \|im\rangle\langle im\| \\ &= \sum_{im} a_i z_m |i\rangle\langle m| \otimes |m\rangle\langle i| \\ &= \sum_{im} a_i z_m R_{im} \otimes R_{im}^\dagger \end{aligned} \tag{7.1}$$

for two (usually distinct) bases of  $H$   $\{|i\rangle\}$  and  $\{|m\rangle\}$ , and accompanying sets of non-negative numbers  $\{a_i\}$  and  $\{z_m\}$  (compare (3.42))? If it can, then  $d$  is a canonical decoherence functional,

$$d_{\alpha\omega}(h, h') = \text{tr}[\rho_\omega h^\dagger \rho_\alpha h'], \tag{7.2}$$

with

$$\rho_\alpha = \sum_{i=1}^N a_i |i\rangle\langle i| \tag{7.3}$$

and

$$\rho_\omega = \sum_{m=1}^N z_m |m\rangle\langle m|. \tag{7.4}$$

As  $d(1, 1) = 1$ , the lack of a normalization constant in front of the trace in (7.2) corresponds to the choice

$$\text{tr}_H \rho_\alpha \rho_\omega = 1. \tag{7.5}$$

There are (at least) two ways to answer the question of whether a given  $d$  is canonical, which, moreover, provide a means by which to explicitly reconstruct, up to an overall relative scale, the boundary conditions  $\rho_\alpha$  and  $\rho_\omega$ . Both of them employ the operator

$$D = \sqrt{d^\dagger d} \tag{7.6}$$

$$= |d|_{\underline{\mathcal{H}}}, \tag{7.7}$$

where the second equality serves to emphasize that *all three* operator operations in (7.6) are taken in  $\underline{\mathcal{H}} = H \otimes H$ , and not in  $H \otimes H^*$ .

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take us rather far afield, however, and bring with it a virtual blizzard of new notation, so that it will have to be given elsewhere. This section assumes it is a decoherence functional on  $\mathcal{R}_C(H)$  or  $\overline{\mathcal{R}}_C(H)$  that is being considered.

Having computed  $D$ ,<sup>31</sup> each of the following are necessary and sufficient conditions for a positive decoherence functional  $d$  to be canonical:

i)  $D$  is idempotent in  $\mathcal{H}$ ,

$$D \odot D = D. \quad (7.8)$$

Alternately,

ii)  $\overline{D}$  is a projection operator on  $\mathcal{H}$ ,

$$\overline{D} \odot \overline{D} = \overline{D} \quad (7.9)$$

where

$$\overline{D} \equiv N_D^{-1} D^* \odot D \quad (7.10)$$

$$N_D \equiv \text{tr}_{\mathcal{H}} D^* \odot D. \quad (7.11)$$

Notice that these conditions are (not entirely coincidentally) reminiscent of two of the conditions for an ordinary density matrix to be pure, namely, that a density matrix  $\rho$  is a pure state (i) iff  $\rho$  is a projection, or, (ii) iff  $\rho^2 = \rho$ . (A third condition is that  $\rho$  is pure iff  $\text{tr}_H \rho^2 = 1$ .)

First I will explain what these conditions have to do with canonical decoherence functionals, then I will show why they are true. The first step is to compute  $D$  for the canonical decoherence functional (7.1):

$$\begin{aligned} D_{\alpha\omega}^2 &= d_{\alpha\omega}^\dagger d_{\alpha\omega} \\ &= \sum_{im} \sum_{jn} a_i a_j z_m z_n (R_{im}^\dagger \otimes R_{im})(R_{jn} \otimes R_{jn}^\dagger) \\ &= \sum_{im} \sum_{jn} a_i a_j z_m z_n (|m\rangle\langle i| \otimes |j\rangle\langle n| \otimes |i\rangle\langle m| \otimes |n\rangle\langle j|) \\ &= \sum_{im} a_i^2 z_m^2 |m\rangle\langle m| \otimes |i\rangle\langle i| \\ &= \rho_\omega^2 \otimes \rho_\alpha^2. \end{aligned} \quad (7.12)$$

Thus

$$D_{\alpha\omega} = \rho_\omega \otimes \rho_\alpha, \quad (7.13)$$

or, in history space Dirac notation,

$$D_{\alpha\omega} = \|\rho_\omega\rangle\langle\rho_\alpha|. \quad (7.14)$$

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<sup>31</sup>The only practical method for computing  $D = |d|_{H \otimes H}$  is to diagonalize  $d^\dagger d$  in  $H \otimes H$ ;  $|d|_{H \otimes H}$  is then obtained by taking the square roots of the eigenvalues of  $d^\dagger d$ . The square root of  $d_{\alpha\omega}^\dagger d_{\alpha\omega}$  is easy because it is already diagonal in the basis  $\{|mi\rangle\} = \{|m\rangle \otimes |i\rangle\}$ , cf. (7.12) and (7.32).

A positive decoherence functional  $d$  is canonical if, and only if,  $D = |d|_{\mathcal{H}}$  factors in  $\mathcal{H}$  in this simple way. (7.14) is obviously necessary for  $d$  to be a canonical decoherence functional. That it is also sufficient is proved in section VII C. The canonicity criteria (i) and (ii) are merely necessary and sufficient conditions for the factorization (7.14), as shown in the following two subsections.

Now let me show that  $D_{\alpha\omega}$  satisfies the canonicity conditions (i) and (ii). First, I note that with the chosen normalization  $d_{\alpha\omega}(1,1) = 1 = \langle \rho_\alpha \parallel \rho_\omega \rangle = \text{tr}_H \rho_\alpha \rho_\omega$ , (3.22) or (7.14) shows that  $D_{\alpha\omega} \odot D_{\alpha\omega} = D_{\alpha\omega}$ , so that  $D_{\alpha\omega}$  is idempotent on  $\mathcal{H}$ , with  $\text{tr}_{\mathcal{H}} D_{\alpha\omega} = \langle \rho_\alpha \parallel \rho_\omega \rangle = 1$ . (However,  $D_{\alpha\omega}$  is not a projection on  $\mathcal{H}$  because it is not (generically) self adjoint,  $D_{\alpha\omega}^* \neq D_{\alpha\omega}$ .)  $D_{\alpha\omega}$  thus satisfies condition (i). Similarly,  $\overline{D}_{\alpha\omega} = \frac{\|\rho_\alpha\| \langle \rho_\alpha \parallel \rho_\alpha \rangle}{\langle \rho_\alpha \parallel \rho_\alpha \rangle}$ , which is manifestly a projection on  $\mathcal{H}$ , so that  $D_{\alpha\omega}$  satisfies the second canonicity condition (ii) as well.

(As an aside, I note that  $\text{tr}_{\mathcal{H}} D_{\alpha\omega} = \text{tr}_{H \otimes H} d_{\alpha\omega} = \text{tr}_H \rho_\alpha \rho_\omega$ , and that  $\text{tr}_{\mathcal{H}} d_{\alpha\omega} = \text{tr}_{H \otimes H} D_{\alpha\omega} = \text{tr}_H \rho_\alpha \text{tr}_H \rho_\omega$ . The pattern displayed here does not, however, carry over to the case of a general operator. It may be that satisfaction of these equations is an additional necessary and sufficient condition for  $d$  to be canonical, analogous to the condition that a density matrix is pure iff  $\text{tr}_H \rho^2 = 1$ , but that has not been shown.)

Thus the conditions (i) and (ii) are necessary for  $d$  to be canonical, though it is hardly obvious that they are sufficient. To see why they are, let us first see how these conditions come about. This will also show us how to recover  $\rho_\alpha$  and  $\rho_\omega$  from a canonical  $D$ .

## B. Origin of Canonicity Criteria and Reconstruction of the Boundary Conditions

The root of the (independent) conditions (7.8) or (7.9) for a decoherence functional to be canonical lies in the following observation: given some operator  $B$ , there exist unit vectors  $|b\rangle$  and  $|\beta\rangle$  such that  $B = |b\rangle\langle\beta|$  iff  $\tilde{B} = B/\text{tr} B$  is idempotent. Alternately,  $B = |b\rangle\langle\beta|$  iff  $B^\dagger B$  is a one-dimensional projection. Taking care of the case where  $|b\rangle$  and  $|\beta\rangle$  do not have to be unit, an operator  $B$  simply factors iff either (i)  $\tilde{B} = B/N_1$  ( $N_1 = \text{tr} B$ ) is idempotent, or, equivalently, (ii)  $\overline{B} = B^\dagger B/N_2$  is a (one dimensional -  $\text{tr} \overline{B} = 1$ ) projection, where  $N_2 = \text{tr} B^\dagger B$ . (To be totally accurate, stated in this way the first condition carries the additional assumption that  $B$  is not traceless, which occurs when  $\langle b | \beta \rangle = 0$ ; see section VII C. That does not affect us here because of (3.31).)

The proof of these alternative conditions for the factorization of  $B$  is given in the next section, after the proof of the sufficiency (not just necessity) of the canonicity criteria. Here, I show how to reconstruct  $|b\rangle$  and  $|\beta\rangle$ ; this will show how  $\rho_\alpha$  and  $\rho_\omega$  may be recovered from a  $D$  which factorizes as in (7.14).

From a factorizing  $B$ , it is possible to reconstruct  $|b\rangle$  and  $|\beta\rangle$  up to phase by defining vectors  $|\bar{b}\rangle$  and  $|\bar{\beta}\rangle$  through  $\overline{B} = |\bar{\beta}\rangle\langle\bar{\beta}|$  and  $|\bar{b}\rangle = B|\bar{\beta}\rangle/\sqrt{N_2}$  for  $\langle\bar{\beta}|\bar{\beta}\rangle = \langle\bar{b}|\bar{b}\rangle = 1$ . Recalling that  $B = |b\rangle\langle\beta|$ , and with  $|b\rangle = b e^{i\phi}|\bar{b}\rangle$  and  $|\beta\rangle = \beta e^{i\theta}|\bar{\beta}\rangle$  for positive real  $b, \beta$ ,

$$B^\dagger B = b^2 \beta^2 |\bar{\beta}\rangle\langle\bar{\beta}|, \quad (7.15)$$

so that  $\text{tr} \overline{B} = 1$  fixes  $b^2 \beta^2$  to be equal to

$$\text{tr} B^\dagger B = \langle b | b \rangle \langle \beta | \beta \rangle. \quad (7.16)$$

(It is of course no surprise that, beginning only with  $B$ , it is only the product of the norms of  $|b\rangle$  and  $|\beta\rangle$  that is determined.) Checking the consistency of  $B = |b\rangle\langle\beta|$  with  $|\bar{b}\rangle = B|\bar{\beta}\rangle/\sqrt{N_2}$  then fixes  $\phi = \theta$ , which, being arbitrary anyway, is natural to absorb in the arbitrary phase implicit in  $|\bar{\beta}\rangle$ .

The upshot is that given an operator  $B$  for which either  $\tilde{B} = B/\text{tr } B$  is idempotent, or  $\overline{B} = B^\dagger B/\text{tr } B^\dagger B$  is a projection, we may write  $B = |b\rangle\langle\beta|$ , where

$$|b\rangle = \beta^{-2} B|\beta\rangle, \quad (7.17)$$

$$|\beta\rangle = \beta |\bar{\beta}\rangle \quad (7.18)$$

for arbitrary  $\beta \in \mathbb{R}^+$ , and  $|\bar{\beta}\rangle$  is determined up to a phase (which does not appear in  $B$  anyway) from

$$\overline{B} = |\bar{\beta}\rangle\langle\bar{\beta}|. \quad (7.19)$$

It should by now be clear where this is going. Translating these factorization results into language appropriate for operators on  $\mathcal{H}$ , an operator  $C$  factors,  $C = \|Z\rangle\langle A|$ , iff either (i)  $\tilde{C} = C/\text{tr}_{\mathcal{H}} C$  is idempotent, or, (ii)  $\overline{C}$  is a projection on  $\mathcal{H}$ . ( $\overline{C}$ , of course, is just  $\overline{C}^* \odot \overline{C}$  divided by  $N_C = \text{tr}_{\mathcal{H}} C^* \odot C$ .) If either of these equivalent conditions is satisfied, then as above the factor  $\|A\rangle$  may be reconstructed up to an arbitrary phase as  $\|A\rangle = a\|\bar{A}\rangle$  ( $a \in \mathbb{R}^+$ ), where  $\overline{C} = \|\bar{A}\rangle\langle\bar{A}|$ , and  $\|Z\rangle = a^{-2}C\|A\rangle$ .

Applying this result to an arbitrary  $d$ , the origins of the conditions (7.8) or (7.9) for the “canonicity” of  $d$  are now becoming evident. (Let  $C = D$ , remembering that  $\text{tr}_{\mathcal{H}} D_{\alpha\omega} = \langle\rho_\alpha \| \rho_\omega\rangle = 1$ .) The canonicity conditions are obviously necessary, as  $D_{\alpha\omega}$  obeys them. That they are also sufficient is not as immediately clear, but as I show below, the factorization of  $D = |d\rangle_{H\otimes H}$  as in (7.13) occurs only if  $d$  has the canonical form (7.1). As a bonus, we also know how to explicitly reconstruct the boundary conditions  $\rho_\alpha$  and  $\rho_\omega$  (up to an arbitrary relative scale) from a canonical  $d$ .

### *Another Statement of the Canonicity Criteria*

If all of that is a little too abstract, here is the algorithm with the fancy clothes removed. Diagonalise your positive decoherence functional in  $\mathcal{H}$  as in (3.10) or (7.1), writing this on  $H \otimes H$  as

$$d = \sum_M w_M E_M \otimes E_M^\dagger. \quad (7.20)$$

By the second of the factorization criteria, check to see if the  $\text{tr}_H$  - orthonormal basis  $\{E_M\}$  factors into a pair of orthonormal bases  $\{|i\rangle\}$  and  $\{|m\rangle\}$  of  $H$  (by checking that the  $\{E_M^\dagger E_M\}$  are a set of orthogonal projections on  $H$ , and likewise for the  $\{E_M E_M^\dagger\}$ .) Then set  $E_M^\dagger E_M = |m\rangle\langle m|$  and  $E_M E_M^\dagger = |i\rangle\langle i|$ , so that  $E_M = |i\rangle\langle m|$ . Write now  $(M \equiv (im))$

$$d = \sum_{im} w_{im} |i\rangle\langle m| \otimes |m\rangle\langle i|. \quad (7.21)$$

Now check that the *a priori*  $n = N^2$  ( $\dim H = N$ ) independent numbers  $w_{im}$  factor as well into only  $2N$  independent numbers. This occurs iff  $w_{im}w_{jn} = w_{in}w_{jm}$ , in which case it makes sense to assert that  $w_{im} = a_i z_m$ , where

$$\frac{a_i}{a_j} = \frac{w_{im}}{w_{jm}} \quad (7.22)$$

for any  $m$  and

$$\frac{z_m}{z_n} = \frac{w_{im}}{w_{in}} \quad (7.23)$$

for any  $i$ . This determines the  $a$ 's and  $z$ 's up to an overall relative scale that is set by the requirement that  $\text{tr}_H \rho_\alpha \rho_\omega = 1$ , defining of course  $\rho_\alpha$  and  $\rho_\omega$  through (7.3) and (7.4) up to the same relative scale.

Thus, the property that both the diagonalizing basis *and* the eigenvalues factor is equivalent to the canonical character of any positive decoherence functional on  $\mathcal{R}_C$  ( $\overline{\mathcal{R}}_C$ ).

### C. Proofs

It remains to supply the proofs of the aforementioned factorization conditions, and of the sufficiency of the factorization of  $D$  (as in (7.14)) for the canonicity of a decoherence functional  $d$ . I take the second task first.

It is not at all obvious that the factorization of  $D = |d|_{\mathcal{H}}$  in  $\mathcal{H}$  is sufficient to imply the canonicity of  $d$ , for, in taking the absolute value of  $d$  in  $\mathcal{H} = H \otimes H$ , it would appear that potentially significant information (a “phase”, *i.e.* an  $\mathcal{H}$ -unitary operator; *cf.* (7.27)) has been discarded. Put another way, it is clear that *if*  $D$  factorizes as in (7.14), we may *define* a canonical decoherence functional which reproduces this  $d$ . But might there not be some non-canonical  $d$  which also leads to  $D$ ? Indeed, it is only because  $d$  is self-adjoint in  $\mathcal{H}$ ,  $d = d^*$ , that this cannot occur.

To proceed, it is helpful to employ the “canonical form” of an operator,

$$B = \sum_n b_n |\phi_n\rangle\langle\psi_n|. \quad (7.24)$$

Here, the  $\{|\phi_n\rangle\}$  and the  $\{|\psi_n\rangle\}$  are both orthonormal sets, and the “singular values”  $b_n$  are positive real numbers. In fact, they are the eigenvalues of

$$\begin{aligned} |B| &\equiv \sqrt{B^\dagger B} \\ &= \sum_n b_n |\psi_n\rangle\langle\psi_n|. \end{aligned} \quad (7.25)$$

Having computed  $|B|$  by diagonalizing  $B^\dagger B$ , the  $|\phi_n\rangle$  may then be constructed as  $|\phi_n\rangle = B|\psi_n\rangle/b_n$ . (A proof of (7.24) for any compact – including in particular any finite rank – operator, reasonably obvious in finite dimensions, may be found in [43, section VI.17].)

Defining the operator

$$U = \sum_{\substack{m \\ b_m \neq 0}} |\phi_m\rangle\langle\psi_m|, \quad (7.26)$$

it should be obvious that

$$B = U|B|. \quad (7.27)$$

(This is the “polar decomposition” of the operator  $B$ , analogous to the polar form of a complex number  $z = |z|e^{i\theta}$ ; cf. [43, section VI.4].)

Consider now a canonical decoherence functional (7.1). It may be written in a form convenient on  $\underline{\mathcal{H}} = H \otimes H$  as

$$\begin{aligned} d_{\alpha\omega} &= \sum_{im} a_i z_m |im\rangle\langle mi| \\ &= \sum_N d_N |\phi_N\rangle\langle\psi_N|, \end{aligned} \quad (7.28)$$

making the obvious identifications  $|im\rangle = |i\rangle \otimes |m\rangle$ ,  $N = (im)$ , and

$$|\psi_{im}\rangle = |mi\rangle \quad (7.29)$$

$$|\phi_{im}\rangle = |im\rangle \quad (7.30)$$

$$d_{im} = a_i z_m. \quad (7.31)$$

Writing  $d_{\alpha\omega}$  in terms of the boundary condition eigenbasis  $|im\rangle$  (as in the first line of (7.1)) thus uncovers directly the canonical form (7.24) of  $d_{\alpha\omega}$  in  $\underline{\mathcal{H}}$ .

In a similar fashion, (7.12) reveals

$$\begin{aligned} D_{\alpha\omega} &= |d_{\alpha\omega}|_{\underline{\mathcal{H}}} \\ &= \sum_{im} a_i z_m |mi\rangle\langle mi| \\ &= \sum_N d_N |\psi_N\rangle\langle\psi_N|, \end{aligned} \quad (7.32)$$

so that the polar decomposition of  $d_{\alpha\omega}$  in  $\underline{\mathcal{H}}$  reads

$$d_{\alpha\omega} = U_{\alpha\omega} D_{\alpha\omega} \quad (7.33)$$

where

$$\begin{aligned} U_{\alpha\omega} &= \sum_N |\phi_N\rangle\langle\psi_N| \\ &= \sum_{\substack{im \\ d_{im} \neq 0}} |im\rangle\langle mi|. \end{aligned} \quad (7.34)$$

Conversely, suppose that  $D$  factors in  $\mathcal{H}$ . It can then be written as in (7.32), from which the  $\{|\psi_N\rangle\}$  and  $\{d_N\}$  (which both obviously factor) can be inferred. But what about the  $\{|\phi_N\rangle\}$ ?

Inspection of (7.28) or (7.33) reveals that *any* choice of the  $\{|\phi_{im}\rangle\}$  would lead to the same  $\sqrt{d^\dagger d}$  as in (7.32), so long as  $\{|\psi_{im}\rangle\}$  and  $\{d_{im}\}$  are chosen as in (7.29) and (7.31). However, this arbitrariness does not occur for decoherence functionals because they are Hermitian on  $\mathcal{H}$ ,  $d = d^*$  (recall (3.10).)



To see this, note from (7.28) that

$$\begin{aligned}
|d_{\alpha\omega}^\dagger|_{\underline{\mathcal{H}}} &= \sqrt{d_{\alpha\omega} d_{\alpha\omega}^\dagger} \\
&= \sum_{im} d_{im} |\phi_{im}\rangle \langle \phi_{im}| \\
&= \rho_\alpha \otimes \rho_\omega \\
&= \|\rho_\alpha\rangle \langle \rho_\omega\|
\end{aligned} \tag{7.35}$$

using (7.30). The second equality in (7.35) shows that for any  $d$  the  $\{|\phi_N\rangle\}$  may be recovered, up to phase, as the eigenvectors (in  $\underline{\mathcal{H}}$ ) of  $|d^\dagger|_{\underline{\mathcal{H}}}$ . The last line, however, shows that  $|d_{\alpha\omega}|_{\underline{\mathcal{H}}} = |d_{\alpha\omega}^\dagger|_{\underline{\mathcal{H}}}^*$ , so that if this condition is satisfied, the  $\{|\phi_N\rangle\}$  may be recovered as well from a  $D$  which factors. Specifically, (7.32) shows that if  $|d|_{\underline{\mathcal{H}}} = \sum_{im} a_i z_m |mi\rangle \langle mi|$ , then  $|d^\dagger|_{\underline{\mathcal{H}}} = |d|_{\underline{\mathcal{H}}}^* = \sum_{im} a_i z_m^* |im\rangle \langle im|$ . Making the choice of phase required by  $|\phi_N\rangle = d|\psi_N\rangle/d_N$  (cf. below (7.24)), this shows that  $|\psi_{im}\rangle = |mi\rangle$  implies  $|\phi_{im}\rangle = |im\rangle$ , as in (7.29) and (7.30), and  $d$  therefore assumes the canonical form (7.1) (cf. (7.28).)

The useful observation at this point is that  $|d|_{\underline{\mathcal{H}}} = |d^\dagger|_{\underline{\mathcal{H}}}^*$  is *guaranteed* for any  $d$  by  $d = d^*$ , so knowing only that  $D$  factors is sufficient to imply the canonicity of  $d$ .

(Checking that  $d = d^* \Rightarrow |d|_{\underline{\mathcal{H}}} = |d^\dagger|_{\underline{\mathcal{H}}}^*$  is a reasonably straightforward calculation. One way to proceed is to use (3.17) to show that  $gg^\dagger = (g^\dagger g)^*$  if  $g = g^*$ , and then to use the canonical form (7.24) of  $g$  in  $\underline{\mathcal{H}}$  to confirm that in this case taking square roots in  $\underline{\mathcal{H}}$  “commutes” with taking adjoints in  $\mathcal{H}$ , i.e. that  $(\sqrt{g^\dagger g})^* = \sqrt{(g^\dagger g)^*}$ .)

What about positivity? While it is true that I have shown that the conditions (7.8) or (7.9) are necessary and sufficient to imply that an arbitrary decoherence functional has the form (7.1), they are *not* sufficient to imply that  $d$  is positive. This is because the phases absorbed into the  $|\phi_N\rangle$  to ensure the singular values are positive are lost in the construction of  $|d|_{\underline{\mathcal{H}}}$  (or  $|d^\dagger|_{\underline{\mathcal{H}}}$ ). We can of course use a  $D$  which factors to *define* a truly positive decoherence functional, but in order to determine whether or not  $d$  is canonical to begin with, we must go back and check whether or not all the eigenvalues  $w_{im} = a_i z_m$  are positive. It is for this reason that the canonicity criteria were stated only for explicitly positive decoherence functionals.

Finally, I end this section with the proof of the factorization conditions.

I treat the second condition first. An operator  $B$  factors into unit vectors,  $B = |b\rangle \langle \beta|$ , iff  $B^\dagger B$  is a (one-dimensional) projection.

The “only if” part is manifest:  $B = |b\rangle \langle \beta|$  implies  $B^\dagger B = |\beta\rangle \langle \beta|$  and  $BB^\dagger = |b\rangle \langle b|$  if  $|b\rangle$  and  $|\beta\rangle$  are unit. Conversely, suppose  $B^\dagger B$  is a one-dimensional projection. Then there is some unit vector  $|\beta\rangle$  (unique up to phase) for which  $B^\dagger B = |\beta\rangle \langle \beta|$ .<sup>32</sup> Define  $|b\rangle = B|\beta\rangle$ . Then  $B = |b\rangle \langle \beta| + R$ , where  $R|\beta\rangle = 0$ . But  $\text{tr } B^\dagger B = 1$ . Taking the trace in a basis in which  $|e_1\rangle = |\beta\rangle$  then quickly reveals that  $\text{tr } R^\dagger R = 0$ . As  $R^\dagger R$  is positive,  $R^\dagger R$ , and hence  $R$ , must be zero.

The first condition is usually easier to use. Namely, a trace 1 operator  $B$  factors iff  $B$  is idempotent,  $B^2 = B$ . (In which case, an *arbitrary*  $B$  factors iff  $B/\text{tr } B$  is idempotent,

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<sup>32</sup>For instance, let  $P = B^\dagger B$  and  $\{|i\rangle\}$  be any orthonormal basis. Then  $|\beta\rangle = \frac{\sum_i P|i\rangle}{\|\sum_i P|i\rangle\|}$  will do.

discounting traceless  $B$ 's. Traceless  $B$ 's which factor are nilpotent, but nilpotent operators - which are always traceless - do not have to factor. This bug does not infect the previous factorization condition, or even this one stated with the assumption that  $B$  is trace 1.)

The easiest way to see this employs again the canonical form (7.24) of  $B$ . Given  $B$  in this canonical form, and supposing  $B^2 = B$ , a short calculation (compute  $\langle \phi_k | B | \psi_l \rangle$ ), shows that for each non-zero  $b_k$ ,  $1 = \sum_k b_k \langle \psi_k | \phi_k \rangle$ . As  $\text{tr } B = 1 = \sum_m b_m \langle \psi_m | \phi_m \rangle$ , only one  $b_k$  can be non-zero for idempotent  $B$ , and trace 1 idempotent  $B$ 's therefore factor. The converse, that trace 1  $B$ 's which factor are idempotent, is trivial.

## VIII. SUMMATION

The aim of this investigation has been to study the basic structures of generalized quantum theory in Hilbert space: the operators which represent quantum histories, and the decoherence functional, the generalization of the (algebraic) quantum state functional which measures both the interference between quantum histories, and the probabilities of those histories when they are consistent.

Generalized quantum mechanics is so useful precisely because it is a very general framework in which to formulate quantum mechanical theories. Its basic postulates are designed to capture one of the most characteristic features of quantum theory: not every history that can be described can be assigned a probability consistently [1,2,14]. This feature, coupled with the intimately related facts that the framework is essentially designed around the superposition principle, and is constructed so that its fundamental predictions are probabilities (however interpreted), are what merit its designation as a kind of quantum theory. The really new element generalized quantum mechanics brings to the theoretical table is the decoherence functional. By providing a measure of interference between quantum histories that is internal to the theory, the decoherence functional allows quantum mechanics to be sensibly applied to closed quantum systems. There is no need to rely on an environment external to a quantum system to “measure” it, thereby determining which of the system's histories may be assigned probabilities. This is terribly important in, for instance, quantum cosmology, where the ambition is to apply the principles of quantum mechanics to the entire universe considered as a single quantum system.

However, the very generality of the basic framework of generalized quantum theory means that there is little physics contained in it beyond these seemingly basic features of quantum mechanics. In this respect it is much like Dirac's transformation theory, which is largely empty of physics until it is supplemented by postulates making the connection between the mathematical elements of the theory and things that can be observed (for instance, identifying operators with physical observables; specifying a theory of the dynamics of those observables – in other words, a Hamiltonian; and naming the physical symmetries of the system.) Nevertheless, the quantum mechanical features encoded by generalized quantum theory are of great interest in and of themselves. They seem close to a minimum of what of quantum physics one might want to retain in efforts to generalize to theories without some of the basic structures on which the formulation of ordinary Hamiltonian quantum mechanics depends (unitary time evolution of a “state at a moment of time”, for example [2].) For quantum systems whose basic observables (whatever they are) can be described by Hilbert space operators, the aim of this work has been to examine just what can be inferred only from

the basic assumptions of generalized quantum theory itself. Such properties are common to any theory which shares the same Hilbert space substructure, whatever additional physical assumptions are then laid on top concerning such matters as the nature of the dynamics or the boundary conditions.

Attention has been concentrated here on the two main strategies for representing quantum histories in Hilbert space: the representation of “history propositions” by projection operators, and, for theories with a time, the representation of histories by class operators. The feature these representations on a (finite dimensional) Hilbert space  $H$  have in common is that the linear completion of the collection of history operators is the full space of linear operators on  $H$ ,  $\mathcal{H} = \mathcal{L}(H)$ . An arbitrary decoherence functional extends uniquely to an Hermitian form on  $\mathcal{H}$  [8], thereby making available the highly developed arsenal of tools for studying Hermitian operators. For positive decoherence functionals like the “canonical” one which arises in ordinary Hamiltonian quantum mechanics, the decoherence functional may even be interpreted as an inner product on the space of histories.

These identifications make transparent a number of useful properties of decoherence functionals and the consistent sets of histories which they define that are independent of any particular theory. For instance, there is a bound on the maximum number of histories in a consistent set, originating in the observation that histories with non-zero probability must be linearly independent of one another. Positive decoherence functionals (like the canonical one) also have the useful property that they obey Cauchy-Schwarz and triangle inequalities. In addition, the observation that the decoherence functional defines an Hermitian form on histories suggests a familiar Dirac-like formalism that is very useful for general computations. In this language, for example, the important ILS theorem classifying the decoherence functionals on projection lattices emerges naturally, and a version of the theorem that applies to decoherence functionals on class operators appears as well. It is also possible to characterize the canonical decoherence functionals in a very simple way. The techniques used here may also be of some use in studying, for instance, the symmetries of decoherence functionals (*cf.* [53,54]).

In this “geometric” point of view, sets of consistent histories are just orthogonal vectors in the inner product defined by the decoherence functional. While it is very difficult to find all of the physical histories that are consistent according to a given decoherence functional explicitly, it is easy to construct the history *operators* in  $\mathcal{H}$  which satisfy the consistency conditions. The problem of determining a decoherence functional’s consistent sets is therefore broken down to the problem of characterizing the collection of operators in  $\mathcal{H}$  which represent physical histories. It is a topic for future research to see if this can be done in a way which lends itself to explicit calculation when quantum histories are represented by “class operators” in  $\mathcal{H}$ .

It is desirable not only to make predictions *given* a decoherence functional, but also to be able to constrain by observations what the decoherence functional (generalized quantum state) might be. A step in this direction is finding all the decoherence functionals according to which a given history or set of histories is consistent. This problem can be formulated in a geometric way, clarifying the strategy for its solution. It is shown here how to explicitly construct the decoherence functionals according to which any given exhaustive set of histories is consistent. The solution to the larger problem of how to construct the decoherence functionals according to which all the members of *collections* of exhaustive sets of histories

are consistent is also briefly outlined. More generally, it is an interesting problem to make use of these results to determine how observations constrain the decoherence functional of a closed system, a question to which I hope to return in a future publication.

There remain, of course, a great many other interesting problems to work out. First, there are a number of mathematical issues to be addressed. As noted already, it would for instance be very helpful to have a better understanding of the structure of the class operator spaces  $\mathcal{R}_C(H)$  and  $\overline{\mathcal{R}}_C(H)$ . In particular, it would be good to have both an explicit method for determining when an element of  $\mathcal{H}$  is a class operator (homogeneous or inhomogeneous), and also an algorithm for constructing the families of histories (2.12) which correspond to each class operator. It would also be very nice to have a practical recipe for determining when a non-positive Hermitian functional on  $\mathcal{H}$  is positive on  $\mathcal{R}$  (when  $\mathcal{R} \neq \overline{\mathcal{R}}_C$ .)

Another problem is to generalize what has been done here to infinite dimensional, separable Hilbert spaces. I have tried to formulate things in a way which suggests what needs to be done in infinite dimensions, but I have not attempted to make that extension. (In this endeavour, the work of Isham *et al.* [3–7], and of Wright [8], should be looked to for more specific guidance.) The most troublesome issue is again the structure of the spaces  $\mathcal{R}$ .

Most importantly, there are many interesting *physical* problems to examine. Some interesting physical/philosophical concerns have been raised about the present minimal form of generalized quantum theory. We of course want quantum theory to describe the quasiclassical universe in which we live. One might therefore want to demand that the decoherence functional of the universe, whatever it is, must predict persistent quasiclassical behaviour when the universe is large [17,39,40]. However, generalized quantum theory may not be up to this task without some additional physical input [21,36–38]. (Perhaps it is worth emphasizing that constraints of this kind are a *good* thing, not bad, because they provide us with important information about what the generalized quantum theory of the universe needs to look like.) An interesting project will be to examine just how much of the important work of Dowker and Kent [21] depends on the specific structure of the canonical decoherence functional itself. (They mostly restrict themselves to the case which arises in ordinary Hamiltonian quantum mechanics,  $\rho_\omega = 1$ .)

Some other interesting physical problems include an examination of the physical significance of some of the choices required in designing a Hilbert space generalized quantum mechanics for theories with a time. There are two questions which come to mind immediately that have some bearing on the present work. The first concerns the choice of admissible coarse grainings. While from the point of view of the quantum mechanics of history it appears arbitrary to restrict to homogeneous coarse grainings, the inhomogeneous coarse grainings do not come about as naturally in older formulations of quantum mechanics. Most interesting for present purposes is Hartle’s observation that the admittance of inhomogeneous coarse grainings would appear to preclude in general the introduction of the familiar notion of “state vector at a moment of time” that is so central to Copenhagen quantum mechanics (*cf.* section II C). A clear enunciation of the physical principles guiding the choice of admissible coarse grainings would be a good thing.

The second question concerns the choice of operator representation of histories. Specifically, what precisely is the physical content of restricting to the class operator representation? While it clearly represents the familiar “quantum branching” or “wave function collapse” of which so much has been written (*cf.* section II C), it takes on a

new quality in generalized quantum theory that warrants closer examination. At least in finite dimensions, the choice of class operator representation also has the interesting feature of drastically reducing the number of histories in a consistent set that can have a non-zero probability of being realized. (This is because the non-zero probability histories in a consistent set must be linearly independent.) What is the physical significance of this?

Of course, there are many other questions one might want to consider. For instance, should decoherence functionals always be taken to be positive? This would be a natural and extremely useful additional requirement, so that (for instance) the zero probability histories which may be needed to make a set of histories exhaustive cannot spoil the consistency of an otherwise consistent set. This is highly desirable from a physical point of view, and I suggest it is argument enough to restrict attention to genuinely positive decoherence functionals. In the case of decoherence functionals on class operators, the requirement of positivity would have the additional, seemingly desirable implication that any decoherence functional on  $\mathcal{R}_C$  would be extendible to  $\overline{\mathcal{R}}_C$  without modification.

Another interesting issue concerns whether we would like sets of disjoint propositions *all at a single moment of time* always to decohere, as they do in ordinary Hamiltonian (generalized) quantum mechanics.<sup>33</sup> Closely related to this is the question of the existence of conservation laws and superselection rules in generalized quantum theory (*cf.* [57,16]), the standard proof of the existence of which appears to depend on this property. We must carefully construct a list of requirements to be imposed on physically allowable decoherence functionals, and then determine the class of functionals which satisfy these requirements. It is even possible that a sufficiently strong list could restrict the decoherence functional of a non-relativistic system to be of the canonical form, perhaps even with  $\rho_\omega = 1$ .

More generally, questions like these raise the larger issue of what features of familiar quantum mechanics are truly essential to a quantum theory of the universe, and which are just, well, familiar [29]. (A common example is the reliance on the notion of “state at a moment of time”, with a corresponding unitary time evolution. It is far from clear that a quantum theory of general relativity, itself a theory of space and time, is best formulated in such terms.) Isham *et al.* have investigated this question from the point of view of quantum logic [3,4], but it is of obvious importance to continue this discussion of principle vigorously. The goal is to provide a theoretical foundation for placing general constraints on the kind of decoherence functionals that we want to consider, a foundation that will complement the more thoroughly studied fundamental issues like the quest for a unified theory of particles and spacetime.

I hope to return to some of these topics in future papers.

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<sup>33</sup>This should be clear from (2.17) with  $\rho_\omega$  set equal to 1, and is the reason the need for something like generalized quantum theory is not more immediately apparent in every day quantum mechanics. The manifestation of this property in the general theory is the fact that consistent class operators are linearly independent.

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## APPENDIX A: RAY-COMPLETENESS OF $\overline{\mathcal{R}}_C(H)$ IN $\mathcal{H}$

In this appendix I sketch briefly how to show that  $\overline{\mathcal{R}}_C(H)$  is “ray-complete” in  $\mathcal{H} = \mathcal{L}(H)$ , that is, contains a vector in every ray of  $\mathcal{L}(H)$ , so long as  $k > 3$ . ( $k$  is the cardinality of the temporal support of the fine grained histories; cf. (2.11).) In fact, I show that  $\mathcal{B}_{\frac{1}{8}}(\mathcal{H}) \subset \overline{\mathcal{R}}_C(H)$ . (As  $k$  increases, it will become apparent below that the size of the ball that we *know* to be filled increases. The largest such ball contained in  $\overline{\mathcal{R}}_C$  is *not* known, but as noted at the end of section II C we know that  $\mathcal{R}_C(H) \subset \mathcal{B}_1(\mathcal{H})$  and that  $\overline{\mathcal{R}}_C(H) \subset \mathcal{B}_{N^k}(\mathcal{H})$ .)

Consider any  $L \in \mathcal{L}(H)$  for which  $\|L\| \leq \frac{1}{8}$ . (With  $L = l_{ij}|i\rangle\langle j|$ ,  $\{|i\rangle\}$  some orthonormal basis for  $H$ ,  $|l_{ij}| \leq \frac{1}{8}$ .) The problem is then to construct an (in general inhomogeneous) history equal to  $L$ . Now, the set  $\{P_i P_\epsilon P_\sigma P_j \mid \forall i, j\}$  ( $P_i = |i\rangle\langle i|$ , and so on) is exclusive (mutually disjoint) irrespective of the choice of  $P_\epsilon$  and  $P_\sigma$ . The question is then, for each choice of  $(i, j)$ , can  $P_\epsilon, P_\sigma$  be found for which  $P_i P_\epsilon P_\sigma P_j = \sum_{ij} l_{ij} |i\rangle\langle j|$  for any  $|l_{ij}| \leq \frac{1}{8}$ ? The answer is yes. The sum of such disjoint homogeneous histories is then actually equal to  $L$ , and  $\overline{\mathcal{R}}_C(H)$  actually fills all of  $\mathcal{B}_{\frac{1}{8}}(\mathcal{H})$  at least. (The norm here is of course the standard operator norm on  $H$ ,  $\|h\| = \sup_{\|x\|_H=1} \|h|x\rangle\|_H$ .)

Consider first the case  $i \neq j$ . Take  $|\epsilon\rangle = a|i\rangle + b|j\rangle + c|\perp\rangle$ , where  $|\perp\rangle$  is a unit vector orthogonal to  $|i\rangle$  and  $|j\rangle$ , and take  $P_\sigma = 1$ . Then  $P_i P_\epsilon P_j = ab^*|i\rangle\langle j|$ . Choosing  $c = 0$ ,  $a = Ae^{i\alpha}$ ,  $b = Be^{i\beta}$ ,  $\langle\epsilon|\epsilon\rangle = 1$  implies  $ab^* = A\sqrt{1-A^2}e^{i(\alpha-\beta)}$ . Many choices of  $a$  and  $b$  thus can reproduce any  $|l_{ij}| \leq \frac{1}{2}$  (with  $i \neq j$ ). In fact, so long as  $\dim H > 2$ , other choices of  $c|\perp\rangle$  can give  $|\epsilon\rangle$  perpendicular to  $|\epsilon\rangle$ , hence disjoint from it. Adding two such histories together we can actually get any required  $|l_{ij}| \leq 1$ , or even longer. (Alternately, so long as  $k > 3$ , we can put the *same* projection in at different times, with all others but the first ( $P_i$ ) and the last ( $P_j$ ) being 1. In this way many physically disjoint histories produce identical class operators.)

The case  $i = j$  is just a little more difficult because  $P_\sigma = 1$  allows only the real coefficients  $\langle i|P_\epsilon|i\rangle$ . (This is why we need  $k > 3$ .) Pick some  $|\epsilon\rangle = a|i\rangle + d|\perp\rangle$  ( $\langle i|\perp\rangle = 0$ ) and  $|\sigma\rangle = b|i\rangle + c|\perp\rangle$ . Then  $\langle i|P_\epsilon P_\sigma|i\rangle = a(a^*b + cd^*)b^* = A^2B^2 + A\sqrt{1-A^2}B\sqrt{1-B^2}e^{i(\alpha-\beta+\gamma-\delta)} = f + ge^{i\theta}$ . Varying  $A, B$  and  $\theta$  over their allowed ranges,  $A, B \in (0, 1)$  and  $\theta \in (0, 2\pi)$ ,  $\langle i|P_\epsilon P_\sigma|i\rangle = l_{ij}$  sweeps out a region containing the disk of radius  $\frac{1}{8}$  about the origin. As in the case  $i \neq j$ , doing the same with a set of orthogonal  $|\epsilon\rangle$ 's or  $|\sigma\rangle$ 's, or by inserting the same sequence of projections in different time slots, thus allows the expansion in disjoint histories of any operator in (say) the unit ball in  $\mathcal{H}$ .

It is therefore possible to find (in general, many) vectors  $|\epsilon_{ij}\rangle$ ,  $|\sigma_{ij}\rangle$  for which  $\sum_{ij} P_i P_{\epsilon_{ij}} P_{\sigma_{ij}} P_j = L$ , for any  $\|L\| \leq \frac{1}{8}$ ;  $\overline{\mathcal{R}}_C(H)$  fills  $\mathcal{B}_{\frac{1}{8}}(H)$  (at least), and  $\overline{\mathcal{R}}_C(H)$  is ray-complete in  $\mathcal{L}(H)$ . It should be clear that as  $k$  or  $\dim H$  increases the size of the ball we can fill in this way increases. But I have most certainly *not* found, for instance, the largest solid ball still contained in  $\overline{\mathcal{R}}_C(H)$ .

However, the purely homogeneous extension of the class operator representation,  $\mathcal{R}_C$ , is *not* ray-complete. Consider a class operator  $C_\alpha \neq 1$  (cf. (2.14)), assuming without loss of generality that  $P_1 \neq 1$ . Considered as an operator on  $H$ ,  $\text{Ran } C_\alpha = \text{Ran } P_1$  is a proper subspace of  $H$ .  $C_\alpha$  thus cannot be proportional to any operator whose range is all of  $H$ . There are, of course, many such operators; there are open sets (in, say, the usual norm, or uniform operator, topology) of such operators which do not contain any genuine class operators (2.14). This makes it possible to find decoherence functionals over  $\mathcal{R}_C(H) \times$

$\mathcal{R}_C(H)$  which are not positive on all of  $\overline{\mathcal{R}}_C(H)$ . Even assuming the continuity of decoherence functionals (actually unnecessary in finite dimensions, as it is implied by sesquilinearity), the “probability function”  $p(h) = d(h, h)$  can go negative on subsets of operators in  $\overline{\mathcal{R}}_C$  which are separated from genuine homogeneous class operators by open sets. (For example, suppose that  $d$  has a small negative eigenvalue in the direction of the traceless, rank  $N$  ( $\dim H = N$ ), norm 1 operator  $\mathcal{O} = \frac{1}{N-1}(\sum_{i=1}^{N-1} |i\rangle\langle i| + (1-N)|N\rangle\langle N|)$ .  $\mathcal{O}$ , trace-orthogonal to the class operator 1 (the only rank  $N$  homogeneous class operator), is contained in an open set not containing any rank  $N-1$  operators (and hence no other homogeneous class operators).<sup>34</sup> If  $d$ ’s other, positive, eigenvalues are chosen large enough,  $d$  will be positive on  $\mathcal{R}_C(H)$  but not on all of  $\overline{\mathcal{R}}_C(H)$ .)

Thus, as  $\mathcal{R}_P(H) \subset \overline{\mathcal{R}}_P(H) \subset \mathcal{R}_C(H) \subset \overline{\mathcal{R}}_C(H)$ , only assuming positivity on the largest of these spaces,  $\overline{\mathcal{R}}_C(H)$ , is sufficient to *imply* the positivity of the (linear extension of the) decoherence functional on all of  $\mathcal{L}(H)$ , though we are of course free to assume it anyway. Nevertheless, all of the other powerful apparatus for studying Hermitian forms remains available.

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<sup>34</sup>Note that it is the “direction” of  $\mathcal{O}$  that matters, not its “length” (norm). If you like, imagine tracing out a “curve” in  $\mathcal{H}$  on a constant norm surface by varying the eigenvalues, but keeping the norm and rank constant. In this case, the norm is the standard operator norm, which is just given by the largest singular value (*cf.* (7.24)). For  $\mathcal{O}$ , that is  $\|\mathcal{O}\| = \frac{|1-N|}{N-1} = 1$ . Parameterizing a variation of  $\mathcal{O}$ ’s other eigenvalues away from  $\frac{1}{N-1}$  by  $\lambda$  which keeps them away from zero creates a one parameter family of operators  $\mathcal{O}_\lambda$  for which  $d(\mathcal{O}_\lambda, \mathcal{O}_\lambda)$  will be negative on some open interval in  $\lambda$  around  $\lambda = 0$ . ( $\mathcal{O}_0 = \mathcal{O}$ , of course.)



## APPENDIX B: SUMMARY OF NOTATION

$S$	A physical system.
$H_S$	Hilbert space of states of $S$ .
$\mathcal{U}$	Set of all histories of $S$ .
$\mathcal{S}$	An exclusive, exhaustive set of histories of $S$ .
$\bar{\mathcal{S}}$	A coarse graining of the set $\mathcal{S}$ .
$\mathcal{L}(H_S)$	The (bounded) linear operators on $H_S$ .
$h$	A history (2.11) of $S$ , or one of its operator representatives (2.12) or (2.14).
$k$	Number of times $t_i$ in the temporal support $\mathcal{T}_k = \{t_1 < \dots < t_k\}$ of the history (2.11).
$u$	The fully coarse grained history, represented in Hilbert space by the operator 1.
$P_{\beta_i}^i$	Projection onto range of eigenvalues $\beta_i$ of observable $i$ .
$C_\beta$	“Class operator” (2.14) $\in \mathcal{L}(H_S)$ representing the history (2.11) on $H_S$ .
$\mathcal{R}_{\mathcal{P}}$	Projection representation of the space of histories $\mathcal{U}$ : the collection of all fine-grained and homogeneously coarse-grained projection operators (2.12).
$\overline{\mathcal{R}_{\mathcal{P}}}$	The completion of $\mathcal{R}_{\mathcal{P}}$ by inhomogeneous coarse-grainings.
$\mathcal{R}_{\mathcal{C}}$	Class operator representation of the space of histories $\mathcal{U}$ : the collection of all fine-grained and homogeneously coarse-grained class operators (2.14).
$\overline{\mathcal{R}_{\mathcal{C}}}$	The completion of $\mathcal{R}_{\mathcal{C}}$ by inhomogeneous coarse-grainings.
$\overline{\mathcal{R}}$	Either of $\mathcal{R}_{\mathcal{P}}$ or $\mathcal{R}_{\mathcal{C}}$ ; occasionally also implies $\overline{\mathcal{R}}$ .
$\overline{\mathcal{R}}$	Either of $\overline{\mathcal{R}_{\mathcal{P}}}$ or $\overline{\mathcal{R}_{\mathcal{C}}}$ .
$H$	Stands for $H_S$ if quantum histories of $S$ are represented by class operators (2.14), and for $\otimes^k H_S$ if histories are represented by projection operators (2.12). Otherwise a generic (finite dimensional) Hilbert space.
$\mathcal{H}$	$\mathcal{L}(H)$ , the space of (bounded) linear operators on $H$ . Isomorphic to $H \otimes H^*$ .
$\underline{\mathcal{H}}$	$H \otimes H$ . Isomorphic to $\mathcal{H}$ .
$\mathcal{B}_r(\mathcal{H})$	Ball of radius $r$ in some specified norm (usually the standard operator norm) in the space $\mathcal{H}$ .
$\mathcal{P}(H)$	Lattice of projection operators on $H$ .
$\mathcal{H}_{\mathcal{S}}$	span $\mathcal{S}$ , where $\mathcal{S}$ is a set of histories (or history operators).
$d$	A decoherence functional.
$d_{\alpha\omega}$	A canonical decoherence functional (2.17).
$\rho_\alpha$	Initial boundary condition (density operator) appearing in a canonical decoherence function (2.17).
$\rho_\omega$	Final boundary condition (density operator) appearing in a canonical decoherence function (2.17).
$D$	$\sqrt{d^\dagger d}$ (cf. (7.7)).

$\mathcal{S}_d$	An exclusive, exhaustive set of histories which is consistent according to $d$ , <i>i.e.</i> the elements of $\mathcal{S}_d$ are mutually consistent.
$\mathcal{D}_d$	Collection of histories which decohere according to the decoherence functional $d$ , <i>i.e.</i> the collection of histories which appear in at least one consistent set $\mathcal{S}_d$ : $\mathcal{D}_d = \{h \in \mathcal{R} \mid d(h, 1-h) = 0\}$ . $\mathcal{D}_d = \mathcal{O}_d \cap \mathcal{R}$ . Same for $\mathcal{R} \rightarrow \overline{\mathcal{R}}$ .
$\mathcal{C}_d$	Collection of exclusive, exhaustive <i>sets</i> of histories $\mathcal{S}_d$ consistent according to $d$ . $\mathcal{C}_d = \mathcal{Q}_d \cap \{\mathcal{S}_d\}$ .
$\mathcal{O}_d$	The collection of <i>operators</i> in $\mathcal{H}$ which satisfy the consistency condition: $\mathcal{O}_d = \{h \in \mathcal{H} \mid d(h, 1-h) = 0\}$ . $\mathcal{D}_d = \mathcal{O}_d \cap \mathcal{R}$ . (Generalization of $\mathcal{D}_d$ to any operator in $\mathcal{H}$ .) Same for $\mathcal{R} \rightarrow \overline{\mathcal{R}}$ .
$\mathcal{Q}_d$	Collection of exhaustive ( $\sum h = 1_H$ ), mutually consistent <i>sets</i> of operators $h \in \mathcal{H}$ . $\mathcal{C}_d = \mathcal{Q}_d \cap \{\mathcal{S}_d\}$ . (Generalization of $\mathcal{C}_d$ to any operator in $\mathcal{H}$ .)
$\mathcal{N}_d$	Nullspace of $d$ .
$\mathcal{H}_d$	$\mathcal{H}/\mathcal{N}_d$ . $\mathcal{N}_d^\perp \simeq \mathcal{H}_d$ .
$\mathcal{N}_d^\perp$	Orthogonal complement of $\mathcal{N}_d$ . $\mathcal{N}_d^\perp \simeq \mathcal{H}_d$ .
$ i\rangle$	Element of $H$ .
$ A\rangle$	Element of $\mathcal{H}$ .
$ in\rangle$	Element of $\underline{\mathcal{H}} = H \otimes H$ . $\underline{\mathcal{H}}$ is isomorphic to $\mathcal{H}$ .
$g$	Denotes an operator on $\mathcal{H} \simeq H \otimes H^*$ .
$\underline{g}$	Denotes the operator on $\underline{\mathcal{H}} = H \otimes H$ that corresponds to the operator $g$ on $\mathcal{H}$ ; <i>cf.</i> (3.13) and (3.17).
$*$	Adjoint on $\mathcal{H} \simeq H \otimes H^*$ ; <i>cf.</i> (3.21).
$\dagger$	Adjoint on $\underline{\mathcal{H}} = H \otimes H$ defined by (3.19).
$\odot$	An operator product on $H \otimes H$ that corresponds to the natural operator product on $\mathcal{H}$ . Defined through (3.22).

(Further aspects of these notations are discussed in section IIID.)

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